Splitting of operators, alternate directions and paraxial approximations for the 3-D wave equation

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Splitting d'opérateurs, direction alternées et approximations paraxiales de l'équation des ondes 3-D

Splitting of Operators, Alternate Directions and Paraxial Approximations for the 3-D Wave Equation

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Résumé :
Nous construisons dans ce travail une famille nouvelle d'aproximations paraxiales de l'équation des ondes en dimension 3 qui sont bien adaptées à une résolution numérique par une méthode de directions alternées. Ces équations ne souffrent pas d'effets parasites dûs à l'anisotropie et le coût de leur mise en oeuvre numérique est très faible en comparaisonavec les approximations paraxiales classiques.

Mots clés :
Equations des ondes 3D, approximations paraxcole, directions alternées, méthode de splitting, anisotropie.

Abstract :
We design a family of new 3-D paraxial equations well-adapted to numerical resolution by alternate directions method. These equations do not suffer from bad anisotropic effects and the cost for their numerical integration remains cheap in comparison with classical paraxial equations.

Key words :
3D wave equation, paraxial approximations, alternate directions, splitting methods, anisotropy.

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ABSTRACT

We design a family of new 3-D paraxial equations well-adapted to numerical resolution by alternate directions method. These equations do not suffer from bad anisotropic effects and the cost for their numerical integration remains cheap in comparison with classical paraxial equations.

1 Introduction

Paraxial approximations of the wave equation have been introduced as an approximate but efficient tool for the numerical computation of wave propagation in the neighborhood of a privileged space direction. Such equations have been applied in various domains of physics as ocean acoustics [22] or geophysics [6]. This last application is the one we have in mind in this paper, namely the migration technique for oil propecting by seismic exploration. In this particular case, the privileged direction is the vertical z direction. Our purpose in this introduction is not to explain or to justify this technique which is now extensively used in geophysical data processing [3]. Let us just mention here that it can be interpreted as a first approach to the inverse problem [18]. The main step is the downward extrapolation in the subground of a wavefield known at the surface. This problem can be solved with the help of paraxial approximations and will retain our attention in this article. Let us mention that an alternative approach to the use of paraxial approximations is the use of discrete extrapolation operators as presented for instance in ([13], [12], [14]).

The derivation of paraxial approximations in a homogeneous medium, that we shall recall in section 2.1, is rather natural. There exists a infinite hierarchy of equations ordered by their order of approximation. According to a terminology which is classical in geophysics, each equation can be labelled by an angle which describes the cone of propagation directions that are correctly represented. The extension to variable coefficients rests upon physical assumptions and criteria which can lead to different equations [2], [1]. However, as far as one is concerned with computational problems, the main difficulties are present in the homogeneous

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case. That is why we shall restrict ourselves to constant coefficients. The generalization to variable coefficients will be reported to forthcoming studies.

The numerical approach to 3-D migration has been receiving considerable attention in the past fifteen years, the increasing power of computers making it achievable although remaining always very expensive. Claerbout was the first to introduce fifteen degree and forty-five degree type equations for the extrapolation of 2-D seismic data. Numerically, a classical approach is to use a Cranck Nicholson scheme in depth direction and finite differences for the derivatives with respect to the lateral variable. It leads a linear system with a tridiagonal matrix which is easy to solve. For 3-D problems, all these numerical techniques can be used again. The problem is that the linear system is now much more difficult to invert. This explains why, as far as the computational time is concerned, there is a real gap between 2D and 3D migration. Kern, [15], [17], has proposed to use modern iterative methods to solve it. However, these methods seem to be efficient only for high frequency problems. Another way to overcome the difficulty is the use of splitting or alternate directions method. It reduces the problem to a serie of 2-D extrapolations, the ones being in the inline variable y_1, the other ones in the crossline variable y_2. The weakness of this technique is that if it is efficient for fifteen degree paraxial equation, its application to the classical forty-five degree paraxial equation actually fails. In 1983, Brown suggests to modify the forty-five degree equation in order to obtain an equation suitable for splitting. He obtains an equation which is of forty-five degree type in the inline and crossline directions and of fifteen degree type in the diagonal directions, giving rise to undesirable anisotropic effects. In 1987, Li, [19], proposes to add another term to this equation in order to correct the anisotropy. We are here at the heart of the matter this paper is about. We propose in fact to define new paraxial equations, well adapted to alternate direction techniques and without anisotropic artefacts. The novelty being to allow others directions than the usual crossline and inline directions to be candidates for the splitting (note that Ristow, [21], used a similar idea to mild the anisotropic effect turning by forty-five degree the y_1, y_2 directions at each extrapolation step). We will show these new equations to be forty-five or sixty degree accurate as well as numerically integrable at a cost between two and four times of that for the integration of a fifteen degree equation.

The paper is organized as follows. In section 2.1, we recall the definitions of the classical paraxial equations. In sections 2.2 and 2.3, we give the main principles governing the numerical discretization of paraxial equations and introduce the splitting techniques applied to these equations. Section 3 is the most important one. We give the definition, the construction and the analysis of the accuracy properties of the new paraxial equations. Each equation is linked to a certain number of directions for the splitting. We present equations with four, then three, then still more directions. In section 4, we show migration results obtained with some of these equations. The appendix is devoted to the proof of the well-posedness of the downward extrapolation problems associated to our new equations.

2 The classical approach

2.1 Wave propagation with paraxial wave equations

In the sequel, t denotes time, (y_1, y_2, z) are space variables, z is the privileged direction, (y_1, y_2) are the transverse variables. The velocity is set equal to 1.

We begin with a solution of the wave equation in the whole space,

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial y_1^2} - \frac{\partial^2 v}{\partial y_2^2} - \frac{\partial^2 v}{\partial z^2} = 0 \tag{2.1}$$

with appropriate boundary and initial conditions. In the whole space, this solution can be split into two waves, an upgoing wave and a downgoing wave. The terminology up and down refers to the privileged z direction. In many applications one is interested in computing only one of these waves. In this paper, we are interested in the upgoing wave. It is governed by the one way wave equation

$$\frac{d\hat{v}^+}{dz} + \imath\omega\left(1 - \frac{|k|^2}{\omega^2}\right)^{\frac{1}{2}} \hat{v}^+ = 0 \tag{2.2}$$

with

$$\hat{v}^+(k_1, k_2, z, \omega) = \int\int\int v^+(y_1, y_2, z, t)\exp^{\imath(k_1y_1+k_2y_2-\omega t)} dy_1 dy_2 dt \tag{2.3}$$

and

$$\begin{cases} \left(1 - \frac{|k|^2}{\omega^2}\right)^{\frac{1}{2}} = \sqrt{1 - \frac{|k|^2}{\omega^2}} & \text{if } \frac{|k|}{|\omega|} \leq 1 \\ \left(1 - \frac{|k|^2}{\omega^2}\right)^{\frac{1}{2}} = \imath\sqrt{\frac{|k|^2}{\omega^2} - 1} & \text{if } \frac{|k|}{|\omega|} > 1. \end{cases} \tag{2.4}$$

Formula (2.3) can be interpreted as writing the upgoing solution as the superposition of harmonic plane waves which are evanescent for $|k| > |\omega|$ and propagative for $|k| \leq |\omega|$. For such wave $|k|/|\omega|$ is nothing but $\sin\theta$ where θ is the angle between the propagation direction and the vertical z-direction.

A major difficulty with this equation is that it corresponds to a non local pseudodifferential equation and therefore is not very tractable from a computational point of view.
The idea of the paraxial wave equations is to approximate the square root with a polynomial or a rational fraction in (2.2) so that the new equation is a local Partial Differential Equation (PDE) for v^+,

$$\begin{cases} (1 - |x|^2)^{\frac{1}{2}} \longrightarrow (1 - |x|^2)^{\frac{1}{2}}_{app} \\ x = (x_1, x_2), \quad |x|^2 = x_1^2 + x_2^2, \quad x_1 = \frac{k_1}{\omega}, \quad x_2 = \frac{k_2}{\omega}. \end{cases} \tag{2.5}$$

If we privilege waves propagating in directions close to the z-direction, $|x| = |k/\omega|$ can be considered as a small parameter.
The parabolic or 15 degree paraxial equation is based on the approximation

$$(1 - |x|^2)^{\frac{1}{2}} \approx 1 - \frac{|x|^2}{2} \qquad (\text{error}: \ O(|x|^4)) \tag{2.6}$$

which leads to the second order PDE

$$\frac{\partial^2 v^+}{\partial t^2} + \frac{\partial^2 v^+}{\partial t \partial z} - \frac{1}{2}\Delta v^+ = 0 \tag{2.7}$$

$$\text{with } \Delta v^+ = \frac{\partial^2 v^+}{\partial y_1^2} + \frac{\partial^2 v^+}{\partial y_2^2}. \tag{2.8}$$

Higher order Taylor expansions are known to give rise to ill-posed problems.
A more accurate approximation is obtained via the first Padé approximation,

$$(1-|x|^2)^{\frac{1}{2}} \approx \frac{1-\frac{3}{4}|x|^2}{1-\frac{1}{4}|x|^2} \qquad (\text{error}:\ O\left(|x|^6\right)). \tag{2.9}$$

It corresponds to a third order PDE, known as the 45 degree paraxial wave equation,

$$\frac{\partial^3 v^+}{\partial t^3} + \frac{\partial^3 v^+}{\partial t^2 \partial z} - \frac{1}{4}\frac{\partial \Delta v^+}{\partial t} - \frac{3}{4}\frac{\partial \Delta v^+}{\partial z} = 0. \tag{2.10}$$

Figures (2.1) and (2.2) show the accuracy of the two approximations representing the variations of the error $\epsilon(x_1, x_2)$ with

$$\epsilon(x_1,x_2) = \epsilon(x) = (1-|x|^2)^{\frac{1}{2}} - (1-|x|^2)^{\frac{1}{2}}_{app}. \tag{2.11}$$

This kind of figures will be used extensively in the sequel and can easily be understood as follows: the larger the white region (error smaller than 10^{-3}) the better the approximation.

Higher order approximations, as a generalisation to (2.9), have been proposed by Bamberger et al., [1]. They are built on fraction expansions

$$\begin{cases} (1-|x|^2)^{\frac{1}{2}} \approx 1 - \sum_{\ell=1}^{L} \beta_\ell \dfrac{|x|^2}{1-\alpha_\ell |x|^2} \\ \alpha_\ell \geq 0,\ \beta_\ell \geq 0,\ 1 \leq \ell \leq L. \end{cases} \tag{2.12}$$

The number L specifies the degree of the approximation and numbers α_ℓ and β_ℓ are chosen to justify to the best the proposed approximation. Padé approximations, [10], correspond to

$$\begin{cases} \beta_\ell = \dfrac{2}{2L+1}\sin^2\left(\dfrac{\ell\pi}{2L+1}\right) \\ \alpha_\ell = \cos^2\left(\dfrac{\ell\pi}{2L+1}\right). \end{cases} \tag{2.13}$$

This choice gives an error in $O\left(|x|^{4L+2}\right)$ and $L=1$ restores the 45° equation (2.10).
Instead of deriving from (2.12) the polynomial form that could be linked to a very high order PDE, Bamberger et al. have proposed a formulation with auxiliary functions. They interpret (2.12) as a system of $L+1$ PDEs in the (y_1, y_2, z, t) domain,

$$\begin{cases} \dfrac{\partial v^+}{\partial z} + \dfrac{\partial v^+}{\partial t} - \sum_{\ell=1}^{L} \beta_\ell \dfrac{\partial \varphi_\ell}{\partial t} = 0 \\ \dfrac{\partial^2 \varphi_\ell}{\partial t^2} - \alpha_\ell \Delta \varphi_\ell = \Delta v^+ \qquad \ell = 1,\ldots,L. \end{cases} \tag{2.14}$$

System (2.14) describes paraxial wave equations of high order in a homogeneous media. It corresponds to a well posed problem for positive parameters α_ℓ and β_ℓ, see [20].

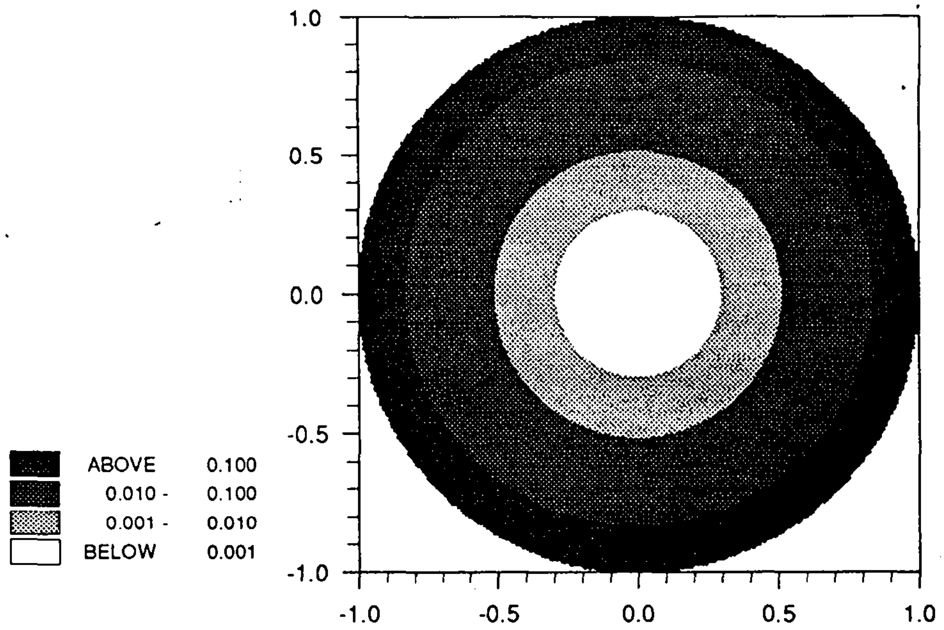

Figure 2.1: Error $\epsilon(x_1, x_2)$ for the 15 degree approximation

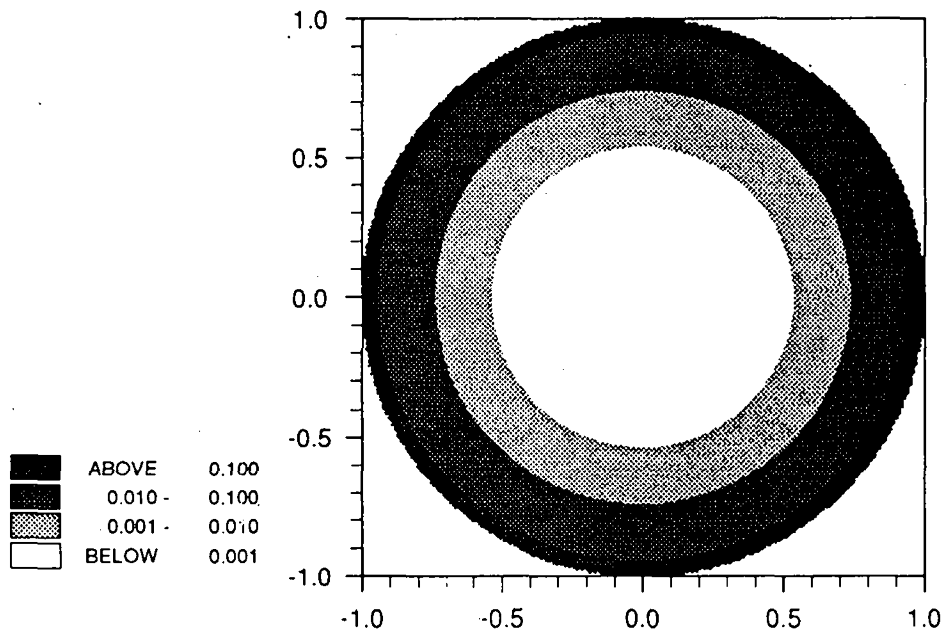

Figure 2.2: Error $\epsilon(x_1, x_2)$ for the 45 degree approximation

2.2 The classical numerical schemes

Numerous authors (e.g. [7], [8]) have proposed numerical schemes for computing the solutions of paraxial equations in two dimensions (i.e., setting $\partial_{y_2} = 0$). The usual numerical schemes are based upon the following principles.

- They work in the frequency domain, i.e. $v^+ \longrightarrow \tilde{v}^+$ with

$$\tilde{v}^+ = \int v^+ \exp^{-\imath\omega t} dt. \tag{2.15}$$

- They handle the transport term along the paraxial axis z exactly, using the change of unknown function $\tilde{v}^+ \longrightarrow u$

$$u = \tilde{v}^+ \exp^{\imath\omega z}. \tag{2.16}$$

- They discretize in y_1 through a variational approach using a uniform mesh and a finite elements/finite differences method.

- They use an implicit and second order accurate Crank-Nicolson scheme for the z direction because classical explicit schemes in z are unstable. z is considered as an evolution variable, ω as a parameter. It can be shown that these schemes are unconditionally stable.

For 3D problems, the same ideas can be reused. In the case of the 15 degree paraxial equation, the equation governing u in the frequency domain is

$$\imath\frac{\partial u}{\partial z} - \frac{1}{2\omega}\Delta u = 0. \tag{2.17}$$

This is a Schrödinger like equation. Following the same principle than for 2D problems, a natural scheme for the z extrapolation is

$$\imath\frac{u^{k+1} - u^k}{\Delta z} - \frac{1}{2\omega}\Delta\left(\frac{u^{k+1} + u^k}{2}\right) = 0. \tag{2.18}$$

Then, using a discrete version of the Laplacian Δ_h on a uniform grid $N \times N$, one has to solve the system (I is identity matrix)

$$\left(I + \frac{\imath\Delta z}{4\omega}\Delta_h\right) u^{k+1} = \left(I - \frac{\imath\Delta z}{4\omega}\Delta_h\right) u^k \tag{2.19}$$

Because of the bad properties (complex, non hermitian, cf Kern [15] and [17]) of the operator $I + \frac{\imath\Delta z}{4\omega}\Delta_h$ this system can be difficult to invert, specially for large N.
In the case of higher order paraxial equations, we encounter the same difficulties: the corresponding equation for u is

$$\begin{cases} \imath\dfrac{\partial u}{\partial z} + \displaystyle\sum_{\ell=1}^{L} A_\ell u = 0 \\ A_\ell = \beta_\ell\omega(-\alpha_\ell\Delta - \omega^2)^{-1}\Delta_h. \end{cases} \tag{2.20}$$

Indeed, to justify rigorously (2.20), we should give a precise meaning to the operator $(-\alpha_\ell\Delta - \omega^2)^{-1}$. This can be done with the help of the limiting absorption principle (see for instance [9]) which is equivalent to use appropriate conditions at infinity. From now on, all the operators of that form will have to be understood in this sense.

Splitting of operators (that we shall describe later) allows us to reduce our presentation to $L = 1$. The system to invert is now (the bar stands for complex conjugate)

$$\begin{cases} (I + d(\omega)\Delta_h)\, u^{k+1} = \left(I - \overline{d(\omega)}\Delta_h\right) u^k \\ d(\omega) = \dfrac{\alpha_1}{\omega^2} + \imath\dfrac{\beta_1 \Delta z}{2\omega}. \end{cases} \tag{2.21}$$

Once again, we come across the same kind of linear system than (2.19).

2.3 Splitting of operators

The splitting of operators [23] is applied to evolution problems like

$$\begin{cases} \imath\dfrac{du}{dz} - Au = 0 \\ u(0) = u_0 \end{cases} \tag{2.22}$$

where we assume that

$$A = A_1 + A_2. \tag{2.23}$$

The exact solution satisfies

$$u(z + \Delta z) = \exp -(\imath A \Delta z) \cdot u(z) \tag{2.24}$$

If A is a bounded operator in a given Hilbert space H, the exponential is defined via the usual series. When A is not bounded but closed with dense domain $D(A)$ and real, that is

$$(Au, u)_H \text{ is real } \quad \forall\, u \in D(A) \tag{2.25}$$

the definition of the exponential follows from the semi-group of operators theory (cf. Kato [16], chapter 9),

$$\exp(-\imath A z) = \lim_{n \to \infty} \left(1 - \imath\frac{zA}{n}\right)^{-n} \quad \text{in } \mathrm{L}\,(H) \tag{2.26}$$

Moreover, $\exp(-\imath A z)$ is an isometry when A is selfadjoint.
If A_1 and A_2 commute, $(A_1 \cdot A_2 = A_2 \cdot A_1)$, one has

$$\exp -(\imath A \Delta z) = \exp -(\imath A_1 \Delta z) \cdot \exp -(\imath A_2 \Delta z)\,. \tag{2.27}$$

otherwise,

$$\exp -(\imath A \Delta z) = \exp -(\imath A_1 \Delta z) \cdot \exp -(\imath A_2 \Delta z) + O\left(\Delta z^2\right). \tag{2.28}$$

This suggests the approximation

$$u^{k+1} = \exp -(\imath A_1 \Delta z) \cdot \exp -(\imath A_2 \Delta z) \cdot u^k \tag{2.29}$$

or, equivalently,

$$\begin{cases} u^{k+\frac{1}{2}} = \exp -(\imath A_2 \Delta z) \cdot u^k \\ u^{k+1} = \exp -(\imath A_1 \Delta z) \cdot u^{k+\frac{1}{2}} \end{cases} \tag{2.30}$$

which leads to the scheme

$$\begin{cases} \imath\dfrac{u^{k+1}-u^{k+\frac{1}{2}}}{\Delta z} &= A_2\left(\dfrac{u^{k+\frac{1}{2}}+u^k}{2}\right) \\ \imath\dfrac{u^{k+1}-u^{k+\frac{1}{2}}}{\Delta z} &= A_1\left(\dfrac{u^{k+\frac{1}{2}}+u^k}{2}\right). \end{cases} \tag{2.31}$$

The resolution of these two systems necessitates to invert successively the two operators $I+\imath\frac{\Delta z}{2}A_1$ and $I+\imath\frac{\Delta z}{2}A_2$. This will be efficient if the inversion of each of these operators is much simpler than the one of $I+\imath\frac{\Delta z}{2}A$.

Remarks

- From the accuracy point of view, splitting methods can be seen as second order accurate, thanks to :,

$$\exp-(\imath A\Delta z)=\exp-(\imath A_1\Delta z/2)\cdot\exp-(\imath A_2\Delta z)\cdot\exp-(\imath A_1\Delta z/2)+O\left(\Delta z^3\right). \tag{2.32}$$

- In the general case (i.e., without any commutation property) this order of convergence seems to be the limit accuracy.
- Of course, splitting methods are generalizable to more than two operators. If A is given by

$$A=\sum_{\ell=1}^{L}A_\ell \tag{2.33}$$

the corresponding splitting algorithm will use L intermediate steps. This is the method we use to approximate system (2.20) by L analogous problems having a unique operator.

In the case of the 15 degree equation (2.17), the splitting

$$\begin{cases} A &= \dfrac{1}{2\omega}\Delta \\ A_1 &= \dfrac{1}{2\omega}\dfrac{\partial^2}{\partial y_1^2} \quad A_2 = \dfrac{1}{2\omega}\dfrac{\partial^2}{\partial y_2^2} \end{cases} \tag{2.34}$$

provides

$$\begin{cases} \imath\dfrac{u^{k+\frac{1}{2}}-u^k}{\Delta z} &= \dfrac{\partial^2}{\partial y_2^2}\left(\dfrac{u^{k+\frac{1}{2}}+u^k}{2}\right) \\ \imath\dfrac{u^{k+1}-u^{k+\frac{1}{2}}}{\Delta z} &= \dfrac{\partial^2}{\partial y_1^2}\left(\dfrac{u^{k+1}+u^{k+\frac{1}{2}}}{2}\right). \end{cases} \tag{2.35}$$

This algorithm corresponds to the so-called alternate directions method. In terms of computational cost, let us consider a finite difference method on a $N\times N$ uniform grid. Using the splitting procedure, one has replaced the resolution of a large but sparse linear system of dimension N^2 with bandwith N (2.19) by 2 families of N independent tridiagonal systems of size N. If we think of direct methods, the cost decreases from $O(N^4)$ to $O(N^2)$. Therefore, the splitting method will be less expensive for large N unless one finds an efficient iterative solver for the large system.

Now, with regard to the 45 degree paraxial equation, the evolution equation to be solved is

$$\left\{\begin{array}{l} \imath \dfrac{du}{dz} = Au \\ A = \dfrac{\omega\Delta}{2}\left(\omega^2 + \dfrac{1}{4}\Delta\right)^{-1}. \end{array}\right. \tag{2.36}$$

The splitting of A is linked to the decomposition of

$$\hat{A}(x_1, x_2) = \frac{x_1^2 + x_2^2}{1 - \frac{1}{4}(x_1^2 + x_2^2)} \tag{2.37}$$

First we note that the decomposition

$$\hat{A}(x_1, x_2) = \frac{x_1^2}{1 - \frac{1}{4}(x_1^2 + x_2^2)} + \frac{x_2^2}{1 - \frac{1}{4}(x_1^2 + x_2^2)} \tag{2.38}$$

has no interest since each term of the sum involves the same operator to be inverted. Secondly, note that replacing $\hat{A}(x_1, x_2)$ by

$$\frac{x_1^2}{1 - \frac{1}{4}x_1^2} + \frac{x_2^2}{1 - \frac{1}{4}x_2^2} \tag{2.39}$$

(as suggested by Brown [4] and used in [11]) provides a nice solution for the splitting of operators but is unfortunatly inconsistant with the 45 degree paraxial equation except for the $y_1 = 0$ and $y_2 = 0$ directions. In the other directions, the approximation is of the same order than for the 15 degree approximation. Figure (2.3) shows the error induced by the corresponding approximation,

$$(1 - |x|^2)^{\frac{1}{2}} \approx 1 - \frac{1}{2}\frac{x_1^2}{1 - \frac{1}{4}x_1^2} - \frac{1}{2}\frac{x_2^2}{1 - \frac{1}{4}x_2^2} \qquad (\text{error}: \ O\left(x_1^2 x_2^2\right)). \tag{2.40}$$

One will note the anisotropy of the result and the loss of accuracy due to the the loss of one order in the approximation for the directions which are not parallel to the axes.

The problem is that there is no exact decomposition of the form

$$\hat{A}(x_1, x_2) = \hat{A}_1(x_1) + \hat{A}_2(x_2) \tag{2.41}$$

and the conclusion of this brief analysis is that no alternate directions method by splitting exists for the 45 degree paraxial equation.

The challenge is then to find a method which gives results whose accuracy is equal to the one given by the 45 degree equation and which cost is comparable to that of the 15 degree equation with splitting. There are at least two ways to do so :

- work on the solver or
- find a new equation.

The first way is studied by Kern ([15], [17]). Kern uses a biconjugate gradient method to solve the large linear system. His hope (that is not reached yet) is that this algorithm will converge in very few iterations with cost $Nb_{iter} \times O(N^2)$, thus becoming competitive with splitting method. We will investigate the second way in the following section.

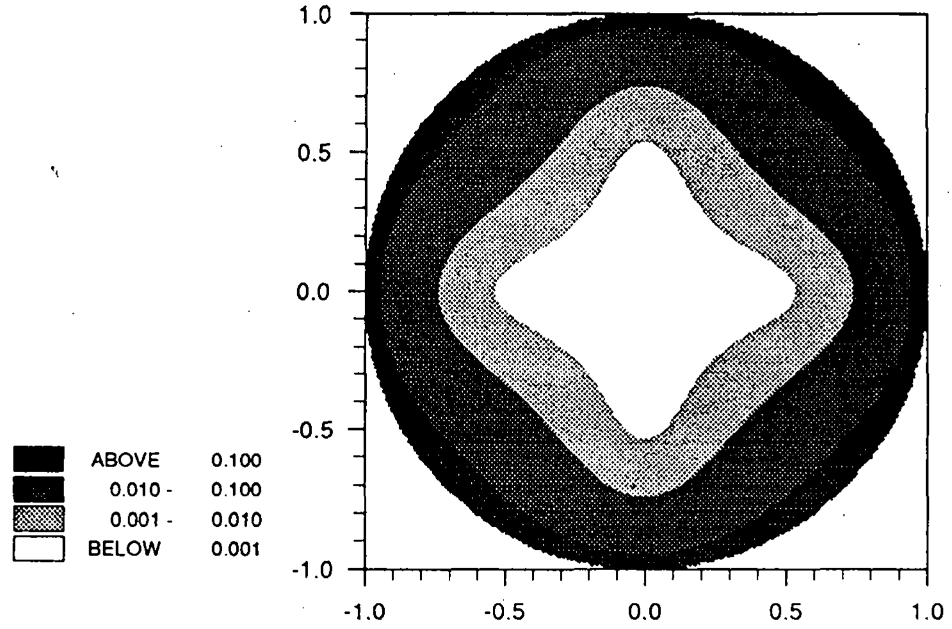

Figure 2.3: Error $\epsilon(x_1, x_2)$ for Brown's approximation

3 Construction of new high order paraxial approximations compatible with splitting methods

3.1 A family of 45 degree approximation with 4 directions

The basic idea is to consider more than two directions for the splitting. The most natural idea is to allow to split the equations along the diagonal directions. As a first step, one has to build new 45 degree type equations which are well adapted to this splitting. We propose to use the following type of approximations, where F, G are rational functions,

$$\left\{ \begin{array}{l} (1 - (x_1^2 + x_2^2))^{\frac{1}{2}} \approx 1 - R(x_1, x_2) \\ R(x_1, x_2) = F(x_1) + F(x_2) + G(x_1 + x_2) + G(x_1 - x_2). \end{array} \right. \tag{3.1}$$

so that one gets an error in $O(|x|^6)$. More precisely and taking into account the symmetries of the problem, we look for an approximation in the form

$$\left\{ \begin{array}{rcl} R(x) & = & b\left(\dfrac{x_1^2}{1 - ax_1^2} + \dfrac{x_2^2}{1 - ax_2^2}\right) + \\ & & +\beta\left(\dfrac{(x_1 + x_2)^2}{1 - \alpha(x_1 + x_2)^2} + \dfrac{(x_1 - x_2)^2}{1 - \alpha(x_1 - x_2)^2}\right). \end{array} \right. \tag{3.2}$$

Because $R(x)$ satisfies

$$\left\{ \begin{array}{l} R(x_1, x_2) = R(x_1, -x_2) = R(-x_1, x_2) \\ R(x_1, x_2) = R(x_2, x_1) \end{array} \right. \tag{3.3}$$

one necessarily has

$$R(x) = A_1(x_1^2 + x_2^2) + A_2(x_1^4 + x_2^4) + Bx_1^2x_2^2 + O\left(|x|^6\right). \tag{3.4}$$

A_1, A_2 and B are (non linear) functions of a, b, α, β

$$A_1 = b + 2\beta, \qquad A_2 = ab + 2\alpha\beta, \qquad B = 12\alpha\beta. \tag{3.5}$$

As we have,

$$\sqrt{1-|x|^2} = 1 - \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{8}(x_1^4 + 2x_1x_2 + x_2^4) + O\left(|x|^6\right) \tag{3.6}$$

we thus obtain

$$b + 2\beta = \frac{1}{2}, \qquad ab + 2\alpha\beta = \frac{1}{8} \qquad \alpha\beta = \frac{1}{48} \tag{3.7}$$

or

$$b + 2\beta = \frac{1}{2} \quad ab = \frac{1}{12} \quad \alpha\beta = \frac{1}{48}. \tag{3.8}$$

This defines a family of 45 degree paraxial approximations depending on one parameter.

From this approximation, we get a new paraxial approximation of (2.2), namely

$$\frac{d\hat{v}^+}{dz} + \imath\omega\left(1 - R\left(\frac{k_1}{\omega}, \frac{k_2}{\omega}\right)\right)\hat{v}^+ = 0, \tag{3.9}$$

which can be written as

$$\begin{cases} \dfrac{d\hat{v}^+}{dz} + \imath\omega\hat{v}^+ - \imath\omega b\hat{\phi}_1 - \imath\omega b\hat{\phi}_2 - \imath\omega\beta\hat{\psi}_+ - \imath\omega\beta\hat{\psi}_- = 0 \\ \hat{\phi}_j = \dfrac{k_j^2}{\omega^2 - ak_j^2}\hat{v}^+ \quad j = 1,2 \quad \hat{\psi}_\pm = \dfrac{(k_1 \pm k_2)^2}{\omega^2 - \alpha(k_1 \pm k_2)^2}\hat{v}^+. \end{cases} \tag{3.10}$$

This corresponds in the (y_1, y_2, z, t) domain to the system of $4L+1$ PDEs,

$$\begin{cases} \dfrac{\partial v^+}{\partial t} + \dfrac{\partial v^+}{\partial z} - b\dfrac{\partial}{\partial t}(\phi_1 + \phi_2) - \beta\dfrac{\partial}{\partial t}(\psi_+ + \psi_-) = 0 \\ \dfrac{\partial^2\phi_1}{\partial t^2} - a\dfrac{\partial^2\phi_1}{\partial y_1^2} = \dfrac{\partial^2 v^+}{\partial y_1^2}, \qquad \dfrac{\partial^2\phi_2}{\partial t^2} - a\dfrac{\partial^2\phi_2}{\partial y_2^2} = \dfrac{\partial^2 v^+}{\partial y_2^2} \\ \dfrac{\partial^2\psi_\pm}{\partial t^2} - \alpha\left(\dfrac{\partial}{\partial y_2} \pm \dfrac{\partial}{\partial y_1}\right)^2\psi_\pm = \left(\dfrac{\partial}{\partial y_2} \pm \dfrac{\partial}{\partial y_1}\right)^2 v^+. \end{cases} \tag{3.11}$$

The equation for u (frequency domain) is given as

$$\begin{cases} \imath\dfrac{du}{dz} + Au = 0 \\ A = A_1 + A_2 + A_+ + A_- \\ A_j = \omega b\left(-\omega^2 - a\dfrac{\partial^2}{\partial x_j^2}\right)^{-1}\dfrac{\partial^2}{\partial x_j^2} \quad j = 1,2 \\ A_\pm = \omega\beta\left(-\omega^2 - \alpha\left(\dfrac{\partial}{\partial y_1} \pm \dfrac{\partial}{\partial y_2}\right)^2\right)^{-1}\left(\dfrac{\partial}{\partial y_1} \pm \dfrac{\partial}{\partial y_2}\right)^2. \end{cases} \tag{3.12}$$

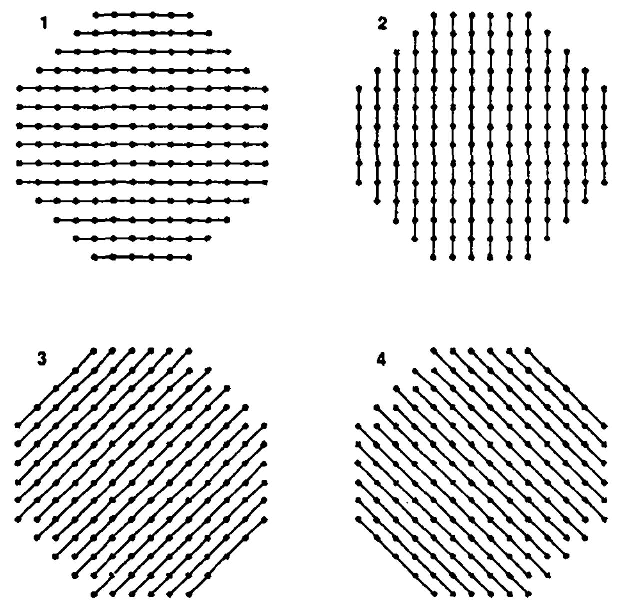

Figure 3.1: Schematic representation of the four steps splitting algorithm. At every step, a tridiagonal system along each lines must be solved.

This paraxial equation can be obviously split into four steps. After standard discretization of the four second derivative operators, one obtains four series of tridiagonal systems as illustrated in figure (3.1). The computational cost is about twice the one of the 15 degree paraxial equation.

In Appendix A, we demonstrate the well-posedness of the new paraxial equation (3.11) with appropriate initial and boundary conditions for positive coefficients a, b, α and β. This result means that the coefficients must be chosen among the positive solutions of system (3.8).

Among the possible choices, one distinguishes

- **The maxi isotropic choice** :

  The 4 space directions play the same role if $a = 2\alpha$ and $b = 2\beta$. In this case, we obtain

$$a = \frac{1}{3}, \quad b = \frac{1}{4}, \quad \alpha = \frac{1}{6}, \quad \beta = \frac{1}{8} \tag{3.13}$$

  and one observes that

$$\sqrt{1 - |x|^2} = 1 - R(x) + \frac{1}{36}|x|^6 + O\left(|x|^8\right). \tag{3.14}$$

  The error is therefore isotropic to the order 8.

- **The Padé approximation in the y_1, y_2 directions** :
  One obtains the usual 45 degree approximation in both directions y_1 and y_2 if $a = \alpha =$

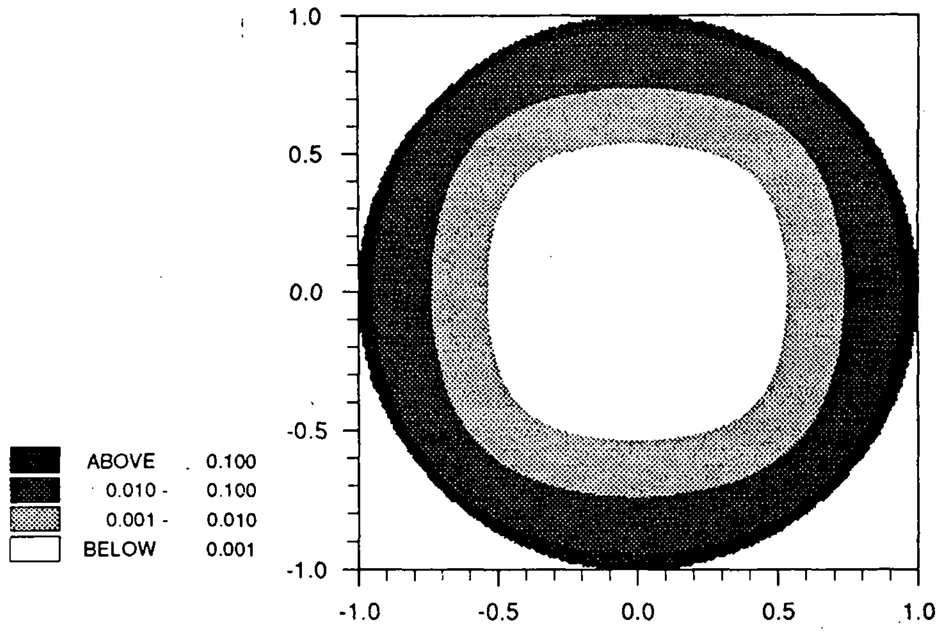

Figure 3.2: Error $\epsilon(x_1, x_2)$ for the 45° approximation restoring Padé in the y_1, y_2 directions

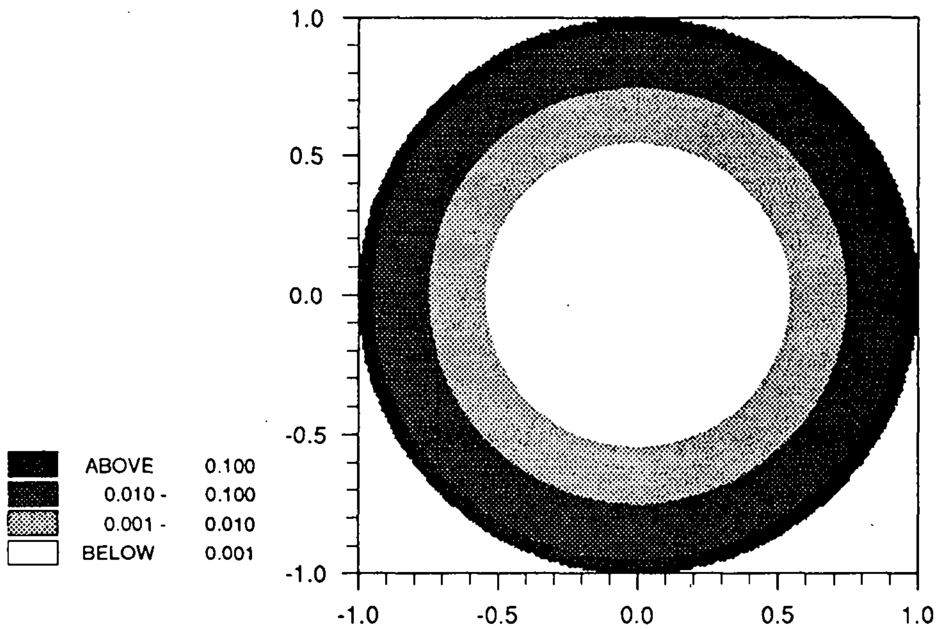

Figure 3.3: Error $\epsilon(x_1, x_2)$ for the maxi isotropic 45° approximation

$1/4$, $b + 2\beta = 1/2$. One obtains

$$a = \frac{1}{4}, \quad b = \frac{1}{3}, \quad \alpha = \frac{1}{4}, \quad \beta = \frac{1}{12} \tag{3.15}$$

Figure (3.3) and (3.2) show the error (2.11) for these two approximations. We observe an improvement of the results with respect to the 15 degree case. The maxi isotropic choice gives a very similar result than for the classical 45 degree approximation. The second choice curiously gives better accuracy in the diagonal directions.

3.2 45 degree approximation with three directions

In the previous section, we have used four directions to obtain an accuracy analogous to those of the usual 45 degree approximation. Here, we propose to obtain the same accuracy but with only three directions. Let us consider the three unit vectors

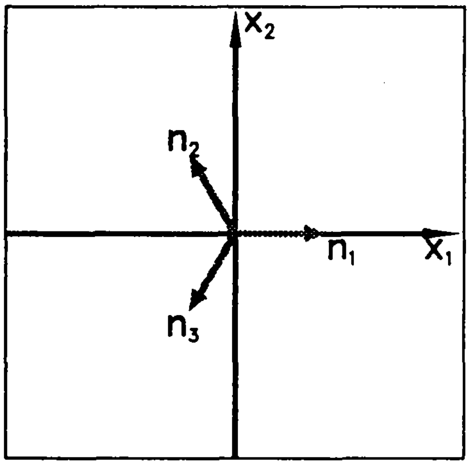

Figure 3.4: The three directions in the (y_1, y_2) plane

$$n_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad n_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad n_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}. \tag{3.16}$$

Now, define

$$\begin{cases} D_1 &= n_1 \cdot \nabla &= \dfrac{\partial}{\partial y_1} \\ D_2 &= n_2 \cdot \nabla &= -\dfrac{1}{2}\dfrac{\partial}{\partial y_1} + \dfrac{\sqrt{3}}{2}\dfrac{\partial}{\partial y_2} \\ D_3 &= n_3 \cdot \nabla &= -\dfrac{1}{2}\dfrac{\partial}{\partial y_1} - \dfrac{\sqrt{3}}{2}\dfrac{\partial}{\partial y_2} \end{cases} \tag{3.17}$$

and set

$$1 - R(x) = 1 - b\sum_{j=1}^{3} \frac{(x \cdot n_j)^2}{1 - a(x \cdot n_j)^2}. \tag{3.18}$$

A Taylor expansion provides

$$R(x) = 1 - b\sum_{j=1}^{3}(x\cdot n_j)^2 - ab\sum_{j=1}^{3}(x\cdot n_j)^4 + O\left(|x|^6\right). \tag{3.19}$$

Further calculation yields

$$\begin{cases} \sum_{j=1}^{3}(x\cdot n_j)^2 &= \frac{3}{2}|x|^2 \\ \sum_{j=1}^{3}(x\cdot n_j)^4 &= \frac{9}{8}|x|^4 \end{cases} \tag{3.20}$$

so that the choice

$$a = \frac{1}{3} \quad b = \frac{1}{3} \tag{3.21}$$

gives an approximate square root to the sixth order.

Proceeding as in the previous section, we get the new paraxial equation as

$$\begin{cases} \frac{\partial v^+}{\partial t} + \frac{\partial v^+}{\partial z} - \frac{1}{3}\frac{\partial}{\partial t}(\phi_1 + \phi_2 + \phi_3) = 0 \\ \frac{\partial^2 \phi_j}{\partial t^2} - \frac{1}{3}D_j^2\phi_j = D_j^2 v^+ \quad j = 1,2,3 \end{cases} \tag{3.22}$$

and the corresponding equation for u is now

$$\begin{cases} \imath\frac{du}{dz} + (A_1 + A_2 + A_3)\,u = 0 \\ A_j = \frac{\omega}{3}D_j^2\left(-\omega^2 - \frac{1}{3}D_j^2\right)^{-1} \quad j = 1,2,3. \end{cases} \tag{3.23}$$

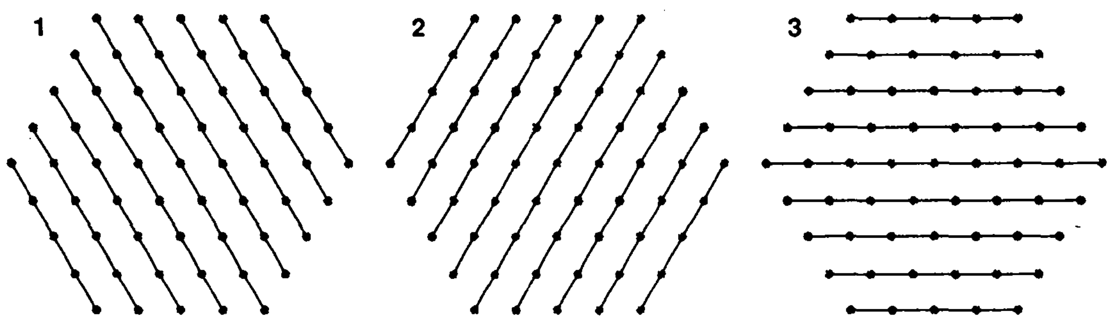

Figure 3.5: Schematic representation of the three steps splitting algorithm. At every step, a tridiagonal system along each lines must be solved.

For the numerical exploitation of this equation, it is natural to use a uniform spatial mesh made of equilateral triangles. Then, using a three steps splitting algorithm and a discrete three points version of the operators D_j^2 allows to compute the solution with a computational cost 1.5 times the one needed for the 15 degree equation.

3.3 Improved approximations with 4 directions

The question we address now is to look if it is possible to get better than 45 degree approximations with 4 directions. We shall see that the answer is yes... up to a certain limit.

We consider the family of approximations

$$\left\{\begin{array}{rcl} \sqrt{1-|x|^2} & \approx & 1-R(x), \\ R(x) & = & \displaystyle\sum_{\ell=1}^{L} b_\ell \left(\frac{x_1^2}{1-a_\ell x_1^2}+\frac{x_2^2}{1-a_\ell x_2^2}\right)+ \\ & + & \displaystyle\sum_{\ell=1}^{L} \beta_\ell \left(\frac{(x_1+x_2)^2}{1-\alpha_\ell (x_1+x_2)^2}+\frac{(x_1-x_2)^2}{1-\alpha_\ell (x_1-x_2)^2}\right) \end{array}\right. \tag{3.24}$$

Using the series expansions

$$\left\{\begin{array}{rcl} R(x) & = & \displaystyle\sum_{n=0}^{+\infty}\left(\sum_{\ell=1}^{L} b_\ell a_\ell^n\right)(x_1^{2n+2}+x_2^{2n+2})+ \\ & & +\displaystyle\sum_{n=0}^{+\infty}\left(\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^n\right)((x_1+x_2)^{2n+2}+(x_1-x_2)^{2n+2}) \\ \sqrt{1-|x|^2} & = & 1-\displaystyle\sum_{n=0}^{+\infty}\gamma_n (x_1^2+x_2^2)^{n+1}, \quad \gamma_n = \left[\begin{array}{l} \frac{1}{2} \text{ if } n=1 \\ \dfrac{1.3.5...(2n-3)}{2^n n!} \text{if } n\geq 2 \end{array}\right. , \end{array}\right. \tag{3.25}$$

the identification of the terms of degree $2n+2$ gives

$$\left\{\begin{array}{l} \displaystyle\left(\sum_{\ell=1}^{L} b_\ell a_\ell^n\right)(x_1^{2n+2}+x_2^{2n+2})+\left(\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^n\right)\sum_{j=0}^{n+1} C_{2n+2}^{2j}\left(x_1^{2j}x_2^{2n+2-2j}+x_2^{2j}x_1^{2n+2-2j}\right) \\ \displaystyle\qquad = \frac{\gamma_n}{2}\sum_{j=0}^{n+1} C_{n+1}^{j}\left(x_1^{2j}x_2^{2n+2-2j}+x_2^{2j}x_1^{2n+2-2j}\right) \end{array}\right. \tag{3.26}$$

or

$$\left\{\begin{array}{l} \displaystyle\left(\sum_{\ell=1}^{L} b_\ell a_\ell^n\right)+2\left(\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^n\right) = \gamma_n \\ \displaystyle 2C_{2n+2}^{2j}\left(\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^n\right) = \gamma_n C_{n+1}^{j}, \quad 1\leq j\leq n. \end{array}\right. \tag{3.27}$$

We see that $n \geq 3$ since $\sum \beta_\ell \alpha_\ell^n$ takes a priori $\left[\frac{n+1}{2}\right]$ values (note that C_n^j is invariant if $j \to n-j$), which means that that (3.27) is impossible if two of these values are different. In particular we can not obtain better than a $O(|x|^8)$ approximation, (i.e., 60° type accuracy) since :

$$2n+2=8 \quad \Leftrightarrow n=3 \tag{3.28}$$

and

$$\frac{C_4^1}{C_8^2}=\frac{C_4^3}{C_8^6}=\frac{1}{7}\neq\frac{3}{35}=\frac{C_4^2}{C_8^4} \tag{3.29}$$

However for $n=2$ we have,

$$
\left\{
\begin{array}{ll}
\displaystyle\sum_{\ell=1}^{L} b_\ell + 2\sum_{\ell=1}^{L} \beta_\ell = \frac{1}{2} & \displaystyle\sum_{\ell=1}^{L} b_\ell a_\ell + 2\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell = \frac{1}{8} \\
\displaystyle\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell = \frac{1}{48} & \displaystyle\sum_{\ell=1}^{L} b_\ell a_\ell^2 + 2\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^2 = \frac{1}{16} \\
\displaystyle\sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^2 = \frac{1}{160}. &
\end{array}
\right.
\tag{3.30}
$$

We have 5 equations. Case $L = 1$ with its four unknowns a_1, b_1, α_1 and β_1 has no solution. Case $L = 2$ is associated to eight unknows and gives a family depending on three parameters. One will distinguish:

- **The maxi isotropic equations**
  They correspond to $a_\ell = 2\alpha_\ell$ and $b_\ell = 2\beta_\ell$, $\ell = 1, 2$. The set of equations becomes

$$
\left\{
\begin{array}{lcl}
b_1 + b_2 & = & \dfrac{1}{4} \\[2mm]
b_1 a_1 + b_2 a_2 & = & \dfrac{1}{12} \\[2mm]
b_1 a_1^2 + b_2 a_2^2 & = & \dfrac{1}{20}
\end{array}
\right.
\tag{3.31}
$$

  which gives a family of equations depending on one parameter. The choice

$$
\left\{
\begin{array}{llllllllllll}
b_1 & = & \dfrac{512}{4341 + 7\sqrt{21705}}, & a_1 & = & \dfrac{13}{32} + \dfrac{\sqrt{21705}}{480}, & \beta_1 & = & \dfrac{b_1}{2}, & \alpha_1 & = & \dfrac{a_1}{2} \\[2mm]
b_2 & = & \dfrac{2293 + 7\sqrt{21705}}{17364 + 28\sqrt{21705}}, & a_2 & = & \dfrac{-93 + \sqrt{21705}}{105 + 3\sqrt{21705}}, & \beta_2 & = & \dfrac{b_2}{2}, & \alpha_2 & = & \dfrac{a_2}{2}
\end{array}
\right.
\tag{3.32}
$$

  gives a $O(|x|^{10})$ error in both x_1, x_2 directions.

- **The Padé approximations in privileged directions**
  There is a possible choice for the coefficients such that our approximation fits the usual 60 degree expansion in direction $y_1 = 0$ and $y_2 = 0$, i.e., the Padé approximation (2.13) with L=2. We obtain

$$
\left\{
\begin{array}{llllllllllll}
b_1 & = & \dfrac{15 + \sqrt{5}}{150}, & a_1 & = & \dfrac{3 + \sqrt{5}}{8}, & \beta_1 & = & \dfrac{45 - 17\sqrt{5}}{600}, & \alpha_1 & = & a_1 \\[2mm]
b_2 & = & \dfrac{-25 + 9\sqrt{5}}{-225 + 75\sqrt{5}}, & a_2 & = & \dfrac{3 - \sqrt{5}}{8}, & \beta_2 & = & \dfrac{25 + 3\sqrt{5}}{900 - 300\sqrt{5}}, & \alpha_2 & = & a_2
\end{array}
\right.
\tag{3.33}
$$

- **The cheap equations**
  They consist in picking $\beta_2 = 0$, which also eliminates α_2. We obtain

$$
\alpha_1 = \frac{3}{10}, \quad \beta_1 = \frac{5}{72} \tag{3.34}
$$

  We then get the set of equations

$$\left\{\begin{array}{lcl} b_1 + b_2 & = & \frac{13}{36} \\ b_1 a_1 + b_2 a_2 & = & \frac{1}{12} \\ b_1 a_1^2 + b_2 a_2^2 & = & \frac{1}{20}. \end{array}\right. \tag{3.35}$$

This, once again, defines a family of equations with one parameter. The choice

$$\left\{\begin{array}{lcllcl} b_1 & = & \frac{1310720}{9321013 + 5177\sqrt{717001}}, & a_1 & = & \frac{989 + \sqrt{717001}}{2560} \\ \beta_1 & = & \frac{5}{72}, & \alpha_1 & = & \frac{3}{10} \\ b_2 & = & \frac{73987249 + 67301\sqrt{717001}}{335556468 + 186372\sqrt{717001}}, & a_2 & = & \frac{-1641 + 3\sqrt{717001}}{5177 + 13\sqrt{717001}} \\ \beta_2 & = & 0, & \alpha_2 & = & 0 \end{array}\right. \tag{3.36}$$

gives a $O(|x|^{10})$ error in both x_1, x_2 directions while the choice

$$\left\{\begin{array}{lcllcl} b_1 & = & \frac{50176}{313209 - 909\sqrt{13385}}, & a_1 & = & \frac{75}{224} - \frac{3\sqrt{13385}}{1120} \\ \beta_1 & = & \frac{5}{72}, & \alpha_1 & = & \frac{3}{10} \\ b_2 & = & \frac{-251709 + 1313\sqrt{13385}}{-1252836 + 3636\sqrt{13385}}, & a_2 & = & \frac{297 + 3\sqrt{13385}}{-505 + 13\sqrt{13385}} \\ \beta_2 & = & 0, & \alpha_2 & = & 0 \end{array}\right. \tag{3.37}$$

gives an isotropic approximation in this family in the sense that the errors in the directions $x_1 = 0$ and $x_1 = x_2$ are equal up to order 10.

Figures (3.6)-(3.9) compare the error for different choices of approximations. Clearly, the maxi isotropic choice gives nicer global results than the isotropic equation with privileged directions. Finally the cheap equation (one saves one auxiliary function) appears as an optimal compromise.

<u>Remark :</u> Most of the coefficients given in this section have been obtained using MAPLE, [5].

3.4 Higher order approximations

To get a better accuracy than $O(|x|^8)$, the above analysis shows that it appears necessary to use more than four directions. This means troubles for the space discretization. Indeed, there is no way of filling the space with a uniform mesh with more than four directions reasonably represented. A possible strategy resorts to

(i) one given reference mesh

(ii) one rotating mesh

(iii) interpolations procedures

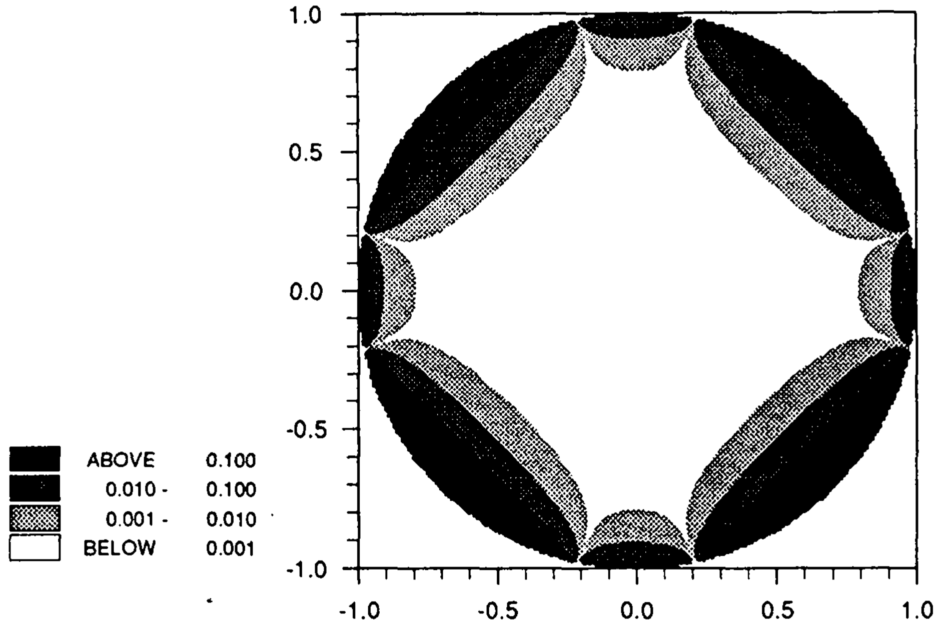

Figure 3.6: Error $\epsilon(x_1, x_2)$ for a 60° approximation restoring Padé in the x_1, x_2 directions

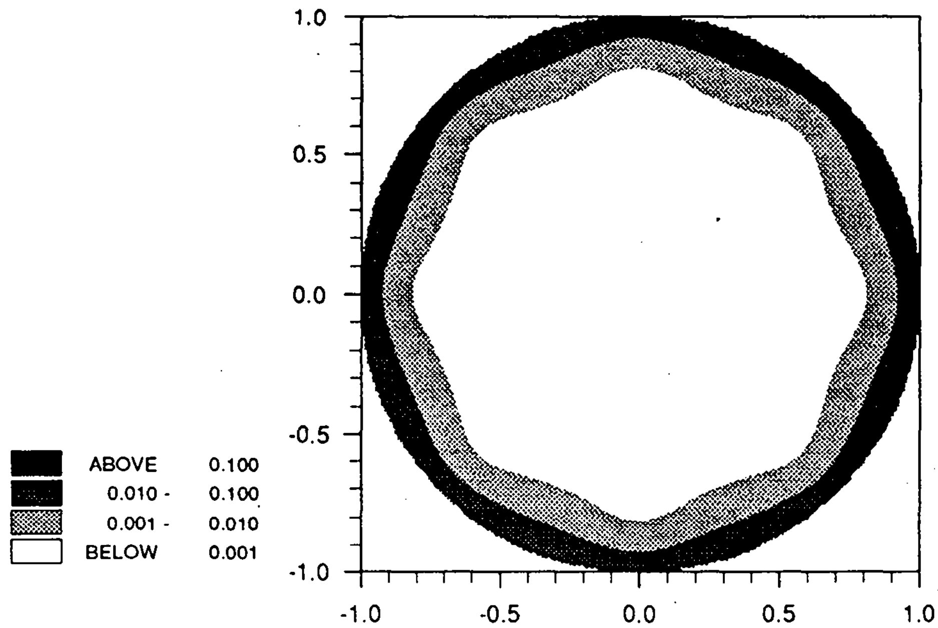

Figure 3.7: Error $\epsilon(x_1, x_2)$ for the maxi isotropic 60° approximation

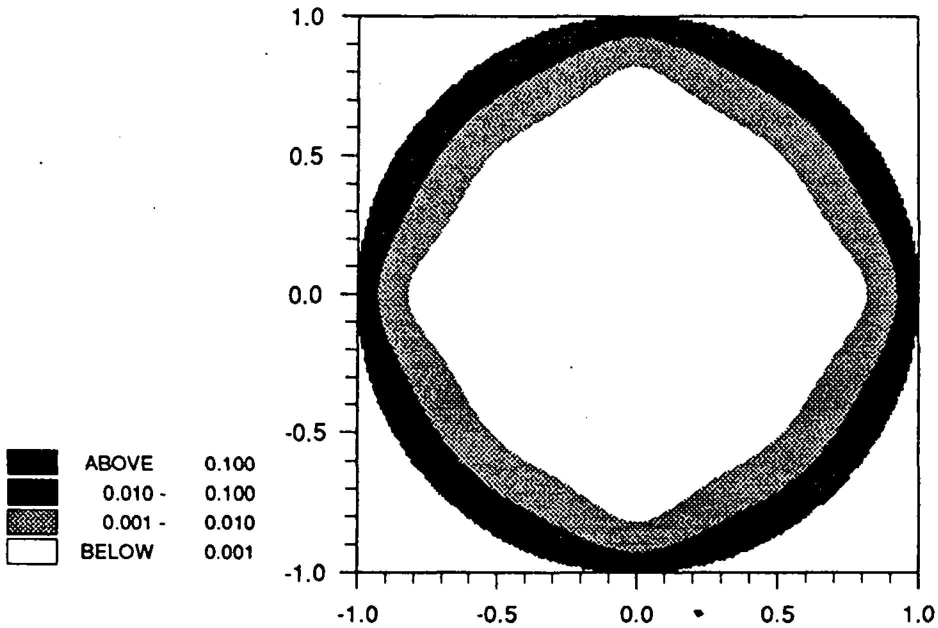

Figure 3.8: Error $\epsilon(x_1, x_2)$ for a cheap 60° approximation with $O(x^{10})$ precision in the y_1, y_2 directions

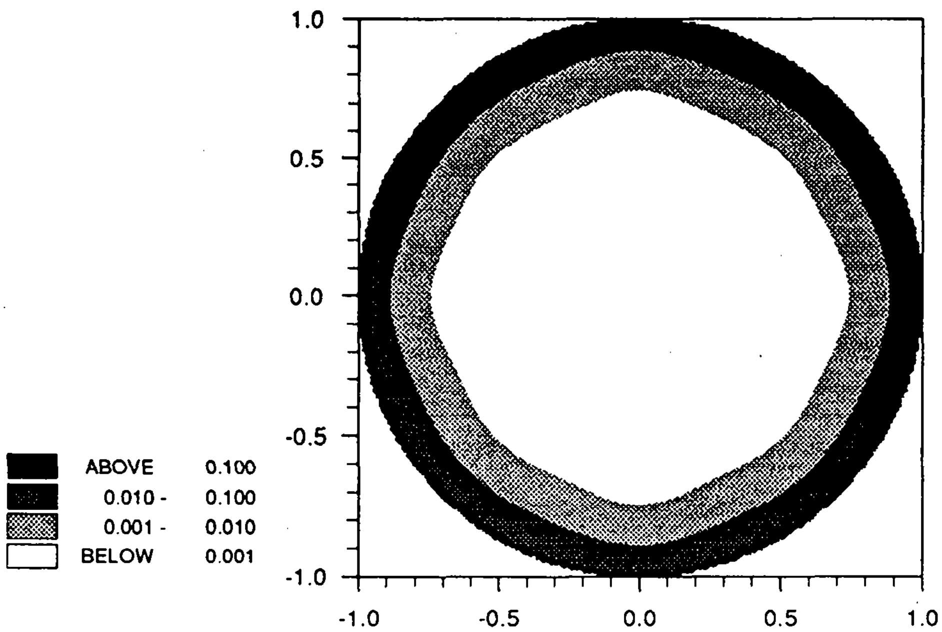

Figure 3.9: Error $\epsilon(x_1, x_2)$ for the isotropic cheap 60° approximation

The algorithm, schematized in figure (3.10), amounts to integrate each step of the splitting and then interpolate the result as many times as necessary.

We construct below a complete family of high order paraxial approximations, which will be in some sense isotropic. Let us consider J angles,

$$0 \leq \theta_j < \pi, \quad j = 1, ..., J \tag{3.38}$$

equally distributed, for instance

$$\theta_j = \frac{(j-1)\pi}{J}. \tag{3.39}$$

Set $n_j = (\cos\theta_j, \sin\theta_j)$ and $D_j = x \cdot n_j$, $j = 1, .., J$. We look for an approximation in the form

$$1 - R(x) = 1 - \sum_{\ell=1}^{L} \beta_\ell \left(\sum_{j=1}^{J} \frac{(x \cdot n_j)^2}{1 - \alpha_\ell (x \cdot n_j)^2} \right). \tag{3.40}$$

The problem is to choose J and $(L, \alpha_\ell, \beta_\ell)$ in such a way that (3.40) approximates $\sqrt{1 - |x|^2}$ up to a given order N. We define polar coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \tag{3.41}$$

and performing series expansion, we obtain

$$1 - R(x) = 1 - \sum_{n=0}^{+\infty} A_n \left(\sum_{j=1}^{J} (x.n_j)^{2n+2} \right), \qquad A_n = \sum_{\ell=1}^{L} \beta_\ell \alpha_\ell^n. \tag{3.42}$$

Using $x \cdot n_j = r\cos(\theta - \theta_j)$, we have

$$1 - R(x) = 1 - \sum_{n=0}^{+\infty} A_n r^{2n+2} \sum_{j=1}^{J} \cos^{2n+2}(\theta - \theta_j). \tag{3.43}$$

By linearization, we get

$$\left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{J} \cos^{2n+2}(\theta - \theta_j) = \Re e \left(\sum_{q=0}^{n+1} \delta_n^q \rho_q \exp^{2iq\theta} \right) \\ \displaystyle\rho_q = \sum_{j=1}^{J} \exp^{-2iq\theta_j}, \quad 0 \leq q \leq n+1 \end{array} \right. \tag{3.44}$$

(δ_n^q denote non zero numbers and $\delta_n^0 = 1$).
If we want to identify error in $O(r^{2N+4})$, we need that

$$\forall q \quad 1 \leq q \leq N+1, \quad \rho_q = \sum_{j=1}^{J} \exp^{-2iq\theta_j} = 0. \tag{3.45}$$

For the choice (3.39), we have

$$\left\{ \begin{array}{l} \displaystyle\rho_q = \frac{1 - \exp^{-2iq\pi}}{1 - \exp^{-2i\pi\frac{q}{J}}} = 0, \quad \text{if } \frac{q}{J} \notin \mathbb{N} \\ \displaystyle\rho_q = J, \quad \text{if } \frac{q}{J} \in \mathbb{N}. \end{array} \right. \tag{3.46}$$

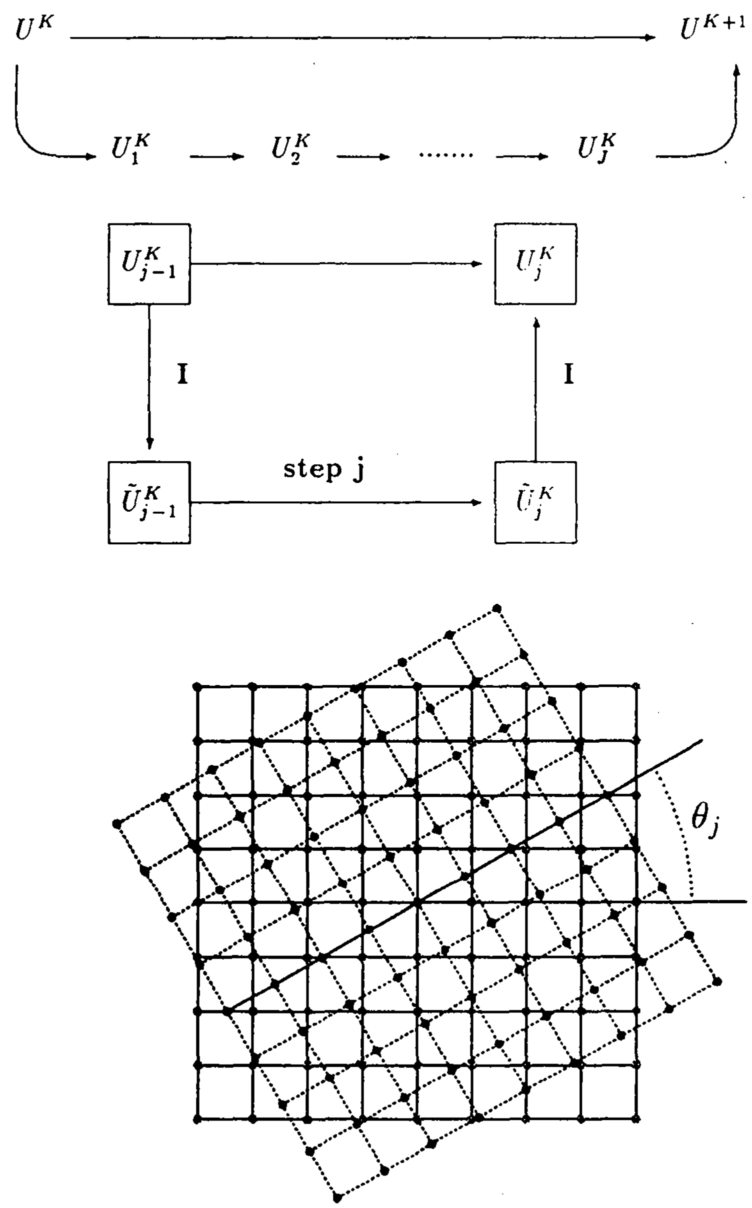

Figure 3.10: Schematic representation of the algorithm

Therefore (3.39) is o.k. for $N+1<J$. This means that in order to get $O(r^{2N+4})$, we need N+2 directions. This result is compatible with those previously obtained: we need 3 directions to get $O(r^6)$ and 4 directions to get $O(r^8)$. Choosing $L=J+2$ and (3.39), we thus obtain

$$1-R(x)=1-\sum_{n=0}^{N}(n+2)A_n|x|^{2n+2}+O\left(|x|^{2N+4}\right) \tag{3.47}$$

to be compared with

$$\sqrt{1-|x|^2}=1-\sum_{n=0}^{N}\gamma_n|x|^{2n+2}+O\left(|x|^{2N+4}\right). \tag{3.48}$$

We get the equations

$$\sum_{\ell=1}^{L}\beta_\ell\alpha_\ell^n=\frac{\gamma_n}{n+2},\quad 0\le n\le N. \tag{3.49}$$

Let α_ℓ and β_ℓ, solutions of (3.49) be given, the corresponding paraxial equation is

$$\left\{\begin{array}{l} \dfrac{d\hat{v}^+}{dz}+\imath\omega\hat{v}^+ -\displaystyle\sum_{\ell=1}^{L}\beta_\ell\sum_{j=1}^{J}\imath\omega\hat{\varphi}_{\ell,j} = 0 \\ \hat{\varphi}_{\ell,j}=\dfrac{(\vec{k}.n_j)^2}{\omega^2-\alpha_\ell(\vec{k}.n_j)^2}\hat{v}^+ \quad j=1,\dots,J,\qquad \ell=1,\dots,L. \end{array}\right. \tag{3.50}$$

Defining the differential operators

$$D_j=n_j.\vec{\nabla}=\cos(\theta_j)\frac{\partial}{\partial y_1}+\sin(\theta_j)\frac{\partial}{\partial y_2} \tag{3.51}$$

and performing inverse Fourier transform, we obtain

$$\left\{\begin{array}{l} \dfrac{\partial v^+}{\partial t}+\dfrac{\partial v^+}{\partial z}-\displaystyle\sum_{\ell=1}^{L}\sum_{j=1}^{J}\beta_\ell\frac{\partial\varphi_{\ell,j}}{\partial t}=0 \\ \dfrac{\partial^2\varphi_{\ell,j}}{\partial t^2}-\alpha_\ell D_j^2\varphi_{\ell,j}=D_j^2v^+,\qquad j=1,\dots,J,\ \ell=1,\dots,L \end{array}\right. \tag{3.52}$$

and the equation for u is

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial z}-\displaystyle\sum_{\ell=1}^{L}\sum_{j=1}^{J}\imath\omega\beta_\ell\psi_{\ell,j}=0 \\ -\omega^2\psi_{\ell,j}-\alpha_\ell D_j^2\psi_{\ell,j}=D_j^2u,\qquad j=1,\dots,J,\ \ell=1,\dots,L. \end{array}\right. \tag{3.53}$$

As we have $2L$ unknowns, we need $N+1\le 2L$. The splitting will have $(N+2)L\sim\frac{N^2}{2}$, $(N\to+\infty)$ intermediate steps.This means that asymptotically the computational cost will increase proportionally to the square of the order of approximation N, while for classical paraxial approximations it is proportional to N. This means that in practise one must not choose N too large if we want our method to be really efficient.

4 Migration of a filtered ponctual source

To compare the different versions of the paraxial equation, we examine the following migration test problem :

$$\left\{ \begin{array}{l} \text{Find } M(y_1, y_2, z) \text{ such that} \\ \\ M(y_1, y_2, z) \;=\; \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \Re e\left(\tilde{w}(y_1, y_2, z, \omega)\right) d\omega \end{array} \right. \tag{4.1}$$

with,

$$\left\{ \begin{array}{l} \tilde{w} \text{ satisfies a one-way wave equation for } z > 0 \\ \\ \tilde{w} \;= \tilde{w}_0 \text{ on } z = 0. \end{array} \right. \tag{4.2}$$

$\tilde{w}_0$, the initial condition at $z = 0$, is a filtered point source, given as

$$\left\{ \begin{array}{l} \tilde{w}_0 \;= \mathrm{F}^{-1}_{y_1, y_2} \left(1_{\{k_1^2 + k_2^2 \leq \omega^2\}}\right) \cdot \tilde{G}(\omega) \\ \\ \mathrm{F}_{y_1, y_2} \text{ denotes the Fourier transform with respect to variables } y_1,\, y_2 \\ \\ \tilde{G}(\omega) = \dfrac{\omega}{\omega_0} \exp^{-\frac{\omega^2}{\omega_0^2}} \exp^{-\imath\omega t_0} \end{array} \right. \tag{4.3}$$

(we have removed the wave numbers associated to evanescent waves).

The reference solution is computed with the exact one-way wave equation

$$\frac{\partial \tilde{w}}{\partial z} + \imath\omega \mathrm{F}^{-1}_{y_1, y_2} \left(1 - \frac{k_1^2 + k_2^2}{\omega^2}\right)^{\frac{1}{2}} \mathrm{F}_{y_1, y_2} \tilde{w} \;=\; 0. \tag{4.4}$$

It gives

$$\tilde{w}(z) \;=\; \mathrm{F}^{-1}_{y_1, y_2} \exp^{\imath\omega z \left(1 - \frac{k_1^2 + k_2^2}{\omega^2}\right)^{\frac{1}{2}}} \mathrm{F}_{y_1, y_2} \tilde{w}_0. \tag{4.5}$$

In figure (5.1), we have represented two slices of the migrated section corresponding to the exact process (4.5). This solution is computed with the following parameters,

$$\left\{ \begin{array}{l} L = \text{ lenght in tranverse directions } y_1,\, y_2 \\ \\ Z = \text{ depth } \;= L/2 \\ \\ t_0 = \text{ time for the explosion } \;= 0.8\, Z \\ \\ F_0 = \dfrac{\omega_0}{2\pi} \;= \text{ characteristic frequency } \;= \dfrac{5}{t_0}. \end{array} \right. \tag{4.6}$$

The space steps Δz in z and h have been chosen equal and such that one has a discretization of 9 points per wavelength in any space direction. This gives 100 points in the directions y_1 and y_2 and 50 points in the direction z.

The first slice is the solution in the plane $y_1 = 0$, whereas the second one depicts the solution in a plane $z = z_0 = 2Z/3$. This depth corresponds to an azimuthal angle θ with

$$\cos(\theta) = (2Z/3) \,/\, (8Z/10) = 20/24 \;\Rightarrow\; \theta = 33.55\ ° \tag{4.7}$$

When using a paraxial equation, the one way wave equation is now

$$\frac{\partial \tilde{w}}{\partial z} + \imath\omega \mathrm{F}^{-1}_{y_1,y_2}\left(1 - R\left(\frac{k_1}{\omega}, \frac{k_2}{\omega}\right)\right)\mathrm{F}_{y_1,y_2}\tilde{w} \;=\; 0, \tag{4.8}$$

where R characterizes the approximation.
The solution is

$$\tilde{w}(z) \;=\; \mathrm{F}^{-1}_{y_1,y_2}\exp^{\imath\omega z\left(1-R\left(\frac{k_1}{\omega},\frac{k_2}{\omega}\right)\right)}\mathrm{F}_{y_1,y_2}\tilde{w}_0 \tag{4.9}$$

In figure (5.2) and (5.3) we compare the reference solution with classical approximate solutions, computed respectively with the 15 and 45 degree Padé approximations and the Brown approximation. Clearly, the 15 degree approximation is not accurate enough to give a satisfactory result. Brown approximation makes appear a lot of anisotropy in the constant depth plane. 45 degree approximation is the only one to give good results. But, as mentionned before, it results to a linear system difficult to invert in the case of heterogeneous media.

In figure (5.4) and (5.5) we compare our reference solution with three approximated solutions corresponding to new paraxial equations. We have picked 45 degree type approximation (3.13) and 60 degree type approximations (3.32) and (3.36). We note that each of the three approximated solutions appears isotropic and is comparable to the solution provided by the classical 45 degree approximation. In other words, we obtain analogous results with equations which can be treated numericaly with alternate directions methods and that are thus much cheaper to solve.

5 Conclusion

We have introduced in this paper new paraxial approximations to the wave equations leading to cheap numerical methods thanks to the help of splitting techniques. The gain with respect to more classical approaches is so important that we may think we could treat with such equations the 3D multishot migration, which was considered up to now as unreachable. We have shown that the the anisotropy which is by nature present present in these new equations can be controlled and reduced by an appropriate choice of coefficients. The first numerical experiments we have made seem to confirm all these good properties. However, the extension to heterogeneous media, as well as a deeper mathematical analysis of the properties of the equations remain to be done. Also, numerical schemes (fourth order schemes for instance, see [15]) have to be investigated and there real efficiency has to be checked. These will be the subjects of our future work in this area.

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References

[1] A. BAMBERGER, B. ENGQUIST, L. HALPERN, AND P. JOLY, *Higher order paraxial approximations for the wave equation*, SIAM J: Appl. Math., (1988), pp. 128–154.

[2] ——, *Paraxial approximations in heterogeneous media*, SIAM J: Appl. Math., (1988), pp. 98–128.

[3] A. BERKHOUT, *Seismic migration*, Elsevier, 1983.

[4] D. L. BROWN, *Applications of operator separation in reflection seismology*, Geophysics, 3 (1983), pp. 288–294.

[5] B. W. CHAR AND AL., *Maple V Language Reference Manual*, Springer Verlag, 1992.

[6] J. F. CLAERBOUT, *Imaging the earth's interior*, Blackwell Scientific Publication co., 1983.

[7] F. COLLINO, *Numerical method for 2-d migration*. Annual Report, PSI Consortium, IFP, Rueil-Malmaison, France, 1993.

[8] A. A. DUBRULLE, *On numerical method for migration in layered media*, Geophysical Prospecting, 56 (1983), pp. 237–264.

[9] D. EIDUS, *The principle of limiting absorption*, Amer. Math. Soc. Transl., 47 (1965), pp. 157–191.

[10] B. ENGQUIST AND A. MAJDA, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp., 31 (1977), pp. 629–651.

[11] P. FROIDEVEAUX, *First result of a 3-d prestack migration program*. Annual Report, PSI Consortium, IFP, Rueil-Malmaison, France, 1990.

[12] D. HALE, *3-d depth migration via mcclellan transformations*, Geophysics, 11 (1991), pp. 1778–1785.

[13] ——, *Stable explicit depth extrapolation of seismic wavefield*, Geophysics, 56 (1991), pp. 1770–1777.

[14] O. HOLBERG, *Towards optimum one-way wave propagation*, Geophysical Prospecting, 36 (1988), pp. 99–114.

[15] P. JOLY AND M. KERN, *Numerical methods for 3-d migration*. Annual Report, PSI Consortium, IFP, Rueil-Malmaison, France, 1990.

[16] T. KATO, *Perturbation theory of linear operators*, Springer Verlag, 1980.

[17] M. KERN, *Numerical methods for the 3-d paraxial wave equation in heterogeneous media,*. Annual Report, PSI Consortium, IFP, Rueil-Malmaison, France, 1991.

[18] P. LAILLY, *The seismic inverse problem as a sequence of before stack migrations*, in Proc. of Conf. on Inverse Scattering, Theory and Applic., SIAM, 1983.

[19] Z. LI, *Compensating finite-diffences errors in 3-d migration and modelling*, Geophysics, 56 (1991), pp. 1650–1660.

[20] L. N.TREFETHEN AND L. HALPERN, *Well posedness of one way equations and absorbing boundary conditions*, Math. Comp., 47 (1986), pp. 437–459.

[21] D. RISTOW, *3-D downward extrapolation of seismic data in particular by finite difference method*, PhD thesis, University of Ultrech, The Netherlands, 1980.

[22] F. TAPPERT, *The parabolic approximation method*, in Wave propagation and underwater acoustics, Lecture Notes in Physics, Vol. 70, J.B. Keller and J. Papadakis, eds., 1977.

[23] R. TEMAM, *Sur la stabilité et la convergence de la méthode des pas fractionnaires*, Ann. Math. Pura. Appl., LXXIV (1991), pp. 191–380.

A Well-posedness of the new paraxial equations

The question we address here concerns the mathematical setting of our new paraxial equations. We want the new equations to have a unique solution and we want to have some ideas about their regularity. For this, we will follow the demonstrations for the classical 2D high order paraxial equation in Bamberger et al., 1988.

The more general form of the approximant for the square root we have investigated in this paper is

$$\sqrt{1-|x|^2} \approx 1 - R(x) = 1 - \sum_{\ell=1}^{L} \beta_l \left(\sum_{j=1}^{J} \frac{(x \cdot n_j)^2}{1 - \alpha_\ell (x \cdot n_j)^2} \right) \tag{A1}$$

where

$$n_j = (\cos(\theta_j), \sin(\theta_j)), \quad \theta_j \in [0, 2\pi[. \tag{A2}$$

The paraxial equation associated with (A1) is

$$\frac{d\hat{v}^+}{dz} + \imath\omega \left(1 - R\left(\frac{k_1}{\omega}, \frac{k_2}{\omega}\right)\right) \hat{v}^+ = 0 \tag{A3}$$

or, equivalently,

$$\left\{ \begin{array}{l} \dfrac{d\hat{v}^+}{dz} + \imath\omega\hat{v}^+ - \displaystyle\sum_{\ell=1}^{L} \beta_\ell \sum_{j=1}^{J} \imath\omega\beta_\ell \hat{\varphi}_{\ell,j} = 0. \\ \hat{\varphi}_{\ell,j} = \dfrac{(\vec{k}.n_j)^2}{\omega^2 - \alpha_\ell (\vec{k}.n_j)^2} \hat{v}^+ \quad j = 1, \ldots, J, \qquad \ell = 1, \ldots, L. \end{array} \right. \tag{A4}$$

Defining the differential operators

$$D_j = n_j.\vec{\nabla} = \cos(\theta_j)\frac{\partial}{\partial y_1} + \sin(\theta_j)\frac{\partial}{\partial y_2} \tag{A5}$$

and performing inverse Fourier transform, we obtain

$$\left\{ \begin{array}{ll} \dfrac{\partial v^+}{\partial t} + \dfrac{\partial v^+}{\partial z} - \displaystyle\sum_{\ell=1}^{L}\sum_{j=1}^{J} \beta_\ell \frac{\partial \varphi_{\ell,j}}{\partial t} = 0 & (a) \\ \dfrac{\partial^2 \varphi_{\ell,j}}{\partial t^2} - \alpha_\ell D_j^2 \varphi_{\ell,j} = D_j^2 v^+, \quad j = 1, \ldots, J, \ \ell = 1, \ldots, L. & (b) \end{array} \right. \tag{A6}$$

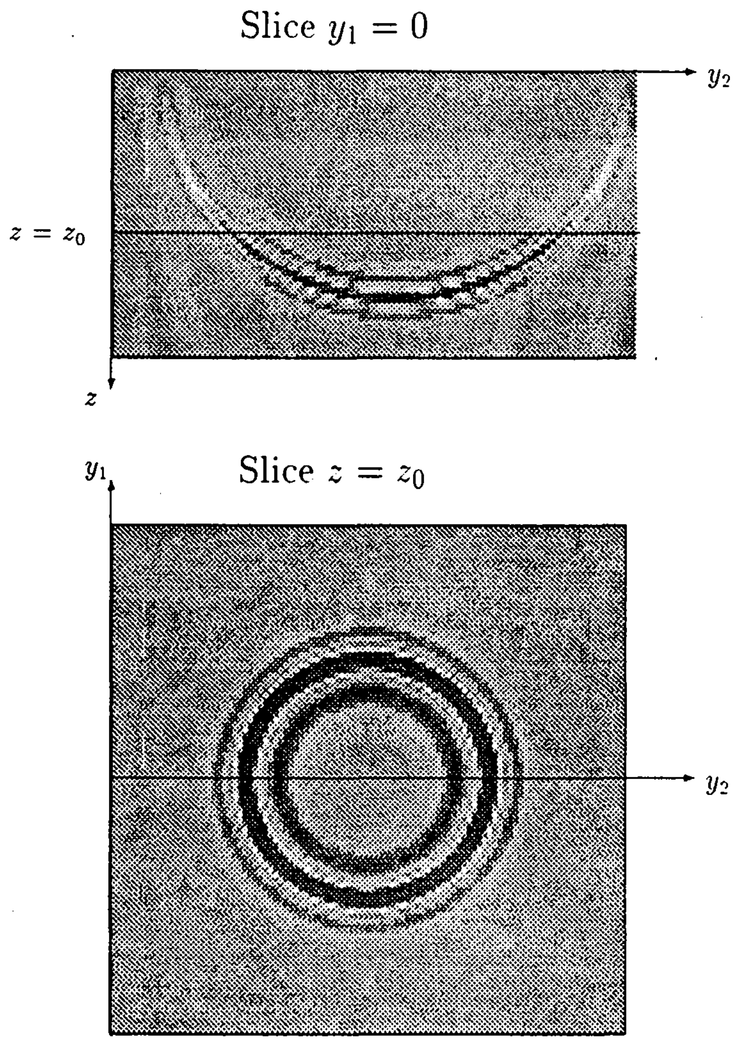

Figure 5.1: Slices $y_1 = 0$ (top) and $z = z_0$ (bottom) for the reference solution

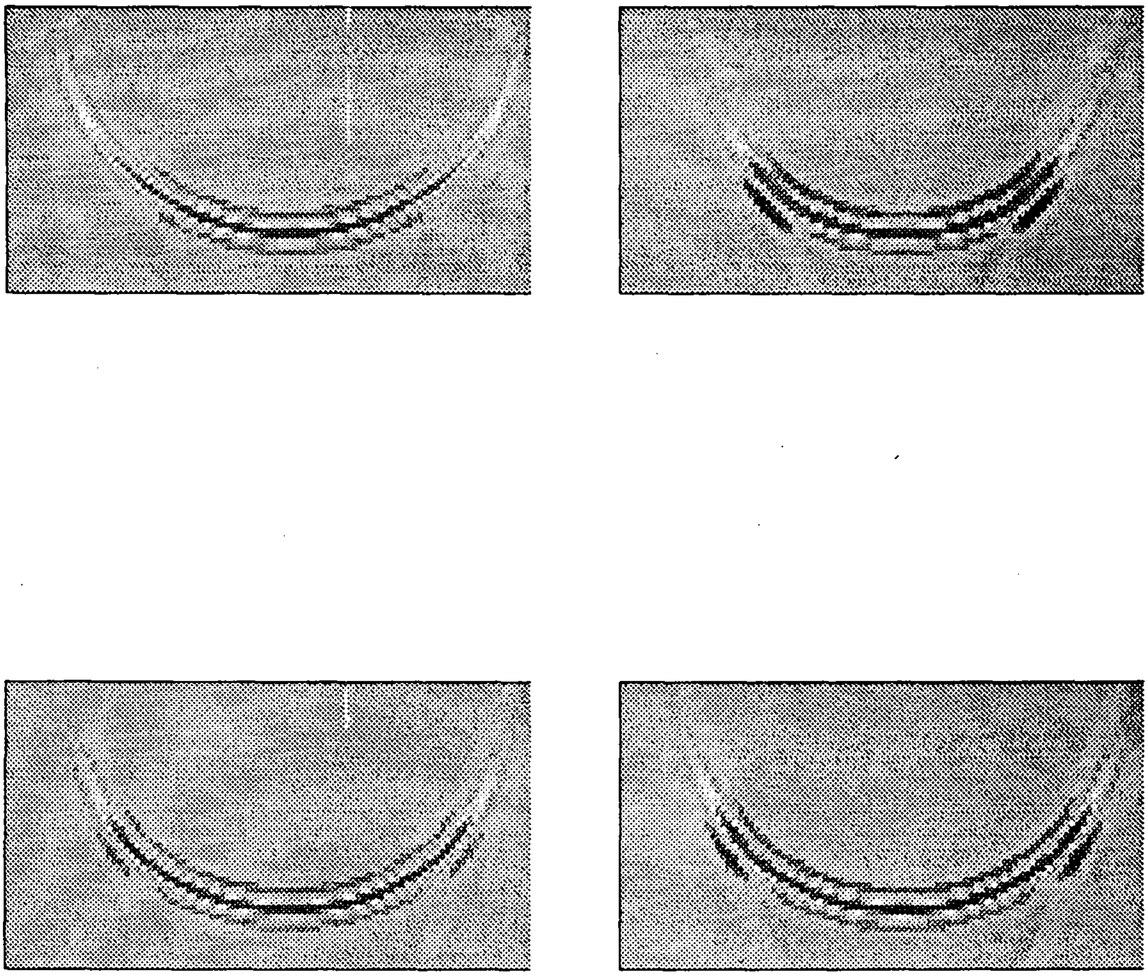

Figure 5.2: Slices at $y_1 = 0$ for the reference solution (top left) and the classical 15 degree (top right), 45 degree (bottom right) and Brown (bottom left) approximations

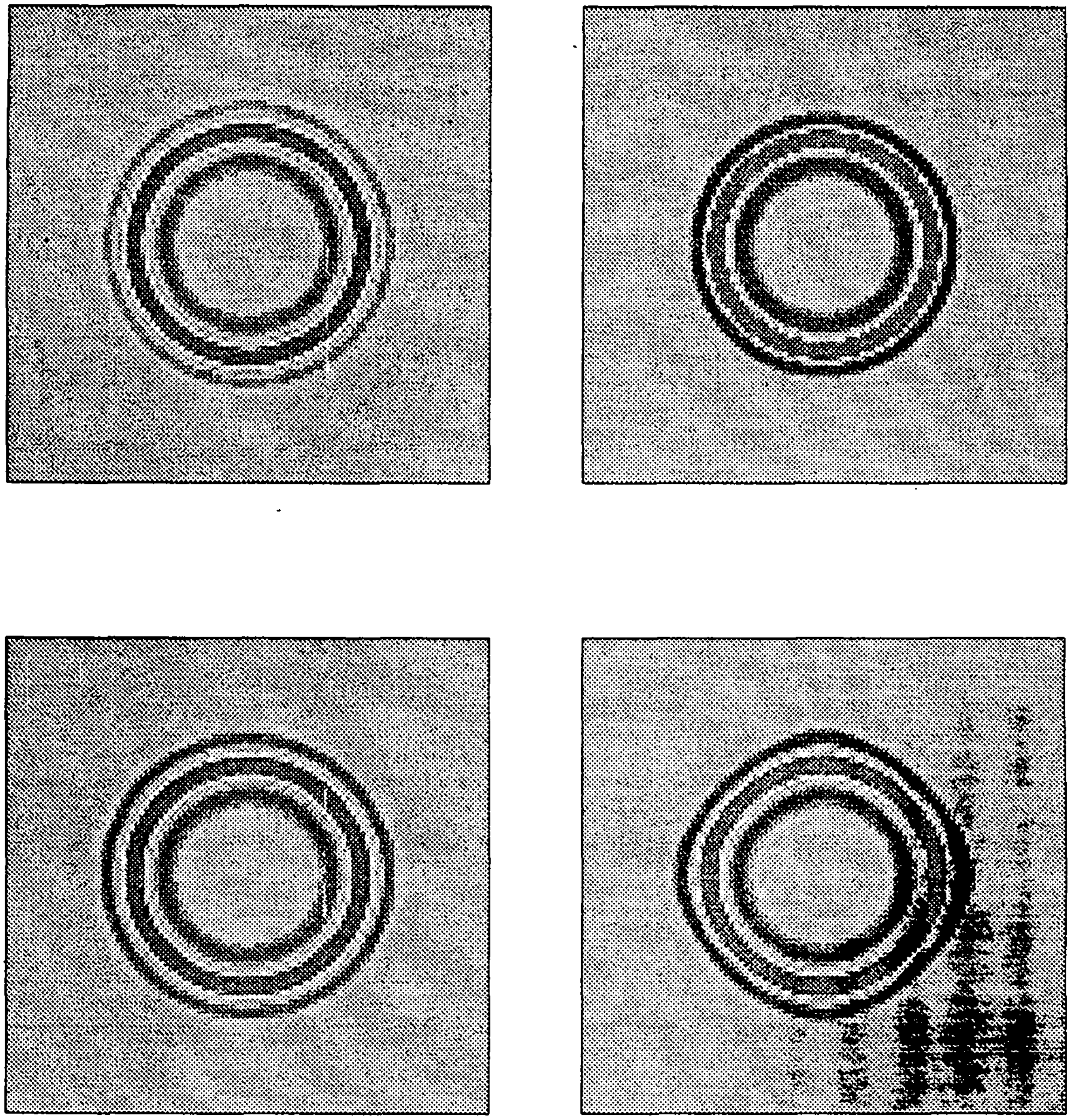

Figure 5.3: Slices at $z = z_0$ for the reference solution (top left) and the classical 15 degree (top right), 45 degree (bottom right) and Brown (bottom left) approximations

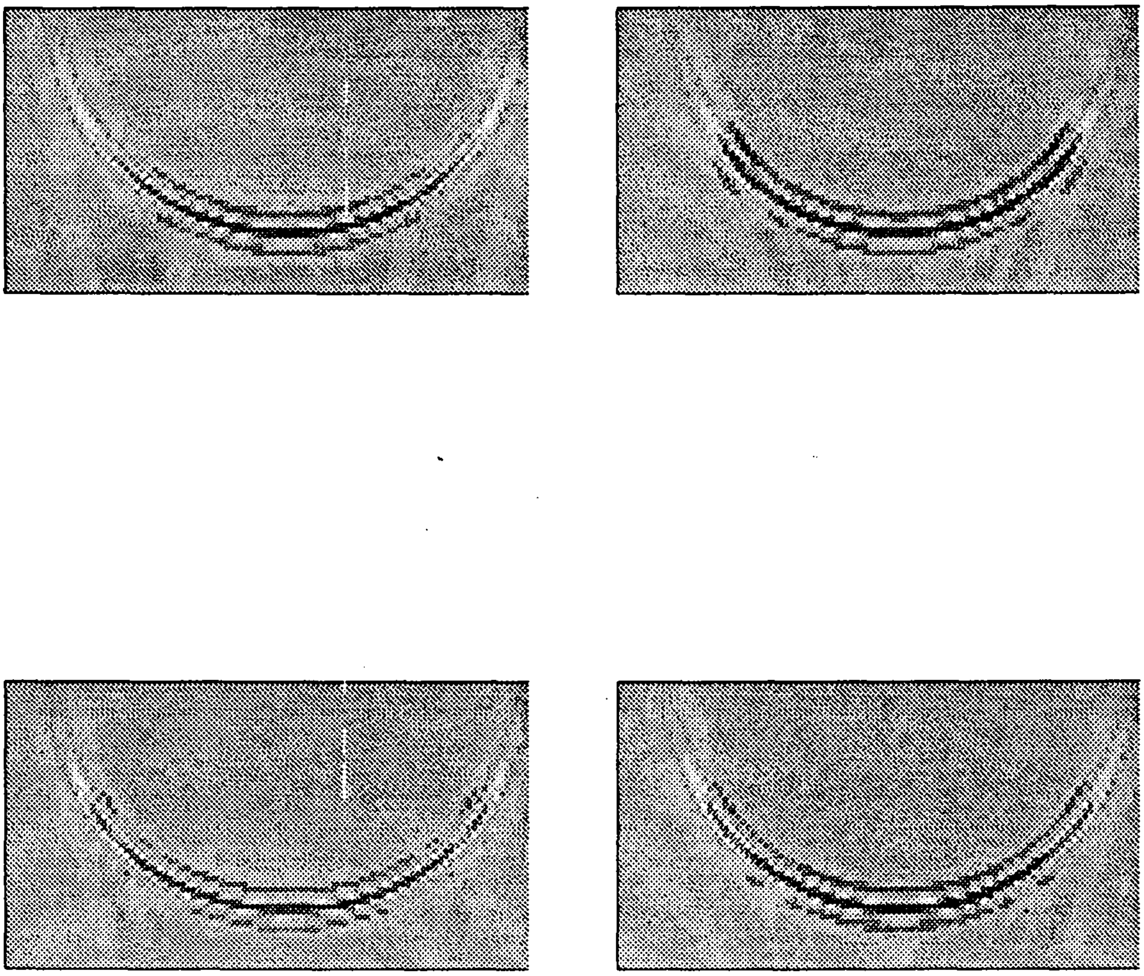

Figure 5.4: Slices at $y_1 = 0$ for the reference solution (top left) and for a new 45 degree (top right) and two new 60 degree (bottom right and left) approximations

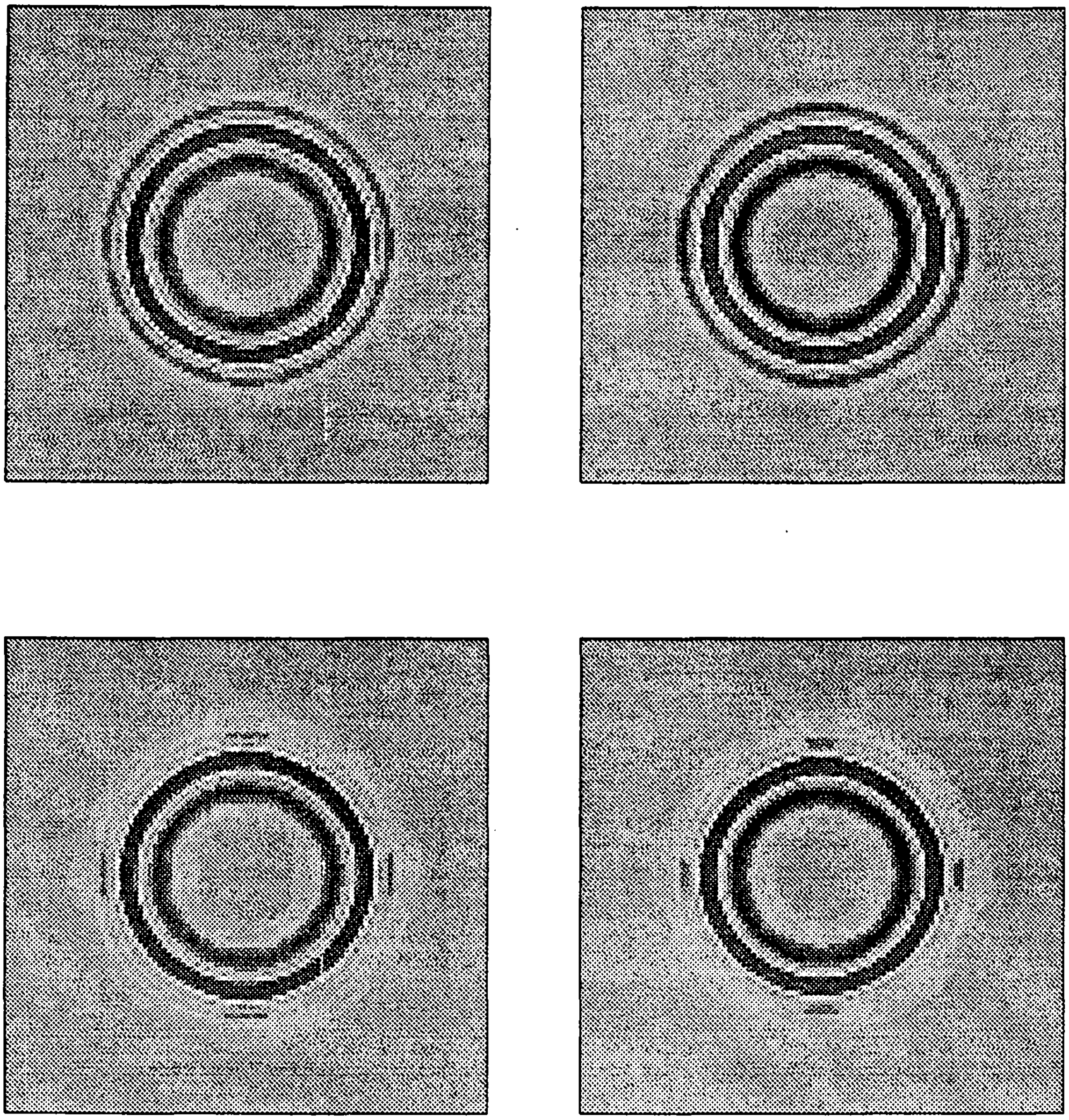

Figure 5.5: Slices at $y_1 = 0$ for the reference solution (top left) and for a new 45 degree (top right) and two new 60 degree (bottom right and left) approximations

We define the initial boundary value problem for (A6) in the half space

$$\Omega = \left\{(y_1, y_2, z) \in \mathrm{R}^3,\ z > 0\right\} \quad \text{as}$$

$$\left\| \begin{array}{l} \textit{Find } (v^+, \varphi_{\ell,j}) \ : \ \Omega \times [0,T] \longrightarrow R \ \textit{ solution of (A6)} \\ \textit{with initial data } \left(v^+, \varphi_{\ell,j}, \partial_t\varphi_{\ell,j}\right)(t=0) = \left(v^0, \varphi^0_{\ell,j}, \varphi^1_{\ell,j}\right) \\ \textit{and a boundary value } v^+(y_1, y_2, 0, t) = g(y_1, y_2, t) \textit{ on } [0,T]. \end{array} \right. \qquad (A7)$$

We establish the following result :

Theorem A.1 *Let α_ℓ, β_ℓ be such that,*

$$\alpha_\ell > 0, \ \ \beta_\ell > 0 \ \ \ell = 1,.., L \qquad (A8)$$

and assume the data to have the regularity

- $v^0, \varphi^0_{\ell,j}, \varphi^1_{\ell,j}, D_j\left(v^0 + \alpha_\ell\varphi^0_{\ell,j}\right) \in L^2(\Omega)$
- $g \in H^1\left(0,T; L^2(\Omega)\right)$,

then, initial boundary value problem (A7) has a unique weak solution with the regularity

- $v^+, \varphi_{\ell,j} \in W^{1,\infty}\left(0,T; L^2(\Omega)\right) \cap W^{2,\infty}\left(0,T; H^{-1}(\Omega)\right)$
- $\dfrac{\partial v^+}{\partial z} \in L^\infty\left(0,T; L^2(\Omega)\right)$
- $D_j\left(v^+ + \alpha_\ell\varphi_{\ell,j}\right) \in L^\infty\left(0,T; L^2(\Omega)\right)$

Moreover, the following energy identity holds :

$$E(t) = E(0) + \frac{1}{2}\int_0^t \int_{z=0} \left|\frac{\partial g}{\partial t}(y_1, y_2, s)\right|^2 dy_1 dy_2 ds \qquad (A9)$$

with $E(t)$ given by

$$E(t) = \frac{1}{2}\int_\Omega \left|\frac{\partial v^+}{\partial t}\right|^2 + \frac{1}{2}\sum_{\ell=1}^{L}\sum_{j=1}^{J}\alpha_\ell\beta_\ell\int_\Omega\left|\frac{\partial \varphi_{\ell,j}}{\partial t}\right|^2 + \frac{1}{2}\sum_{\ell=1}^{L}\sum_{j=1}^{J}\beta_\ell\int_\Omega\left|D_j\left(v^+ + \alpha_\ell\varphi_{\ell,j}\right)\right|^2. \qquad (A10)$$

Proof : We begin to establish the energy identity in three steps. We assume the existence of a solution of problem (A6) in the spaces described above.

i) We differentiate equation (A6.a) with respect to t, multiply by the time derivative of v^+ and integrate over Ω to obtain

$$\int_\Omega \frac{\partial^2 v^+}{\partial t\partial z}\frac{\partial v^+}{\partial t} + \int_\Omega \frac{\partial^2 v^+}{\partial t^2}\frac{\partial v^+}{\partial t} - \sum_{\ell=1}^{L}\beta_\ell\sum_{j=1}^{J}\int_\Omega\frac{\partial^2\varphi_{\ell,j}}{\partial t^2}\frac{\partial v^+}{\partial t} = 0$$

or, equivalently,

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega\left(\frac{\partial v^+}{\partial t}\right)^2\right) - \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v^+}{\partial t}\right)^2 - \sum_{\ell=1}^{L}\beta_\ell\sum_{j=1}^{J}\int_\Omega\frac{\partial^2\varphi_{\ell,j}}{\partial t^2}\frac{\partial v^+}{\partial t} = 0. \qquad (A11)$$

ii) We multiply equation (A6.b) by $\partial_t(v^+ + \alpha_\ell \varphi_{\ell,j})$, integrate over Ω and use an integration by part to obtain

$$\int_\Omega \frac{\partial^2 \varphi_{\ell,j}}{\partial t^2} \frac{\partial v^+}{\partial t} + \alpha_\ell \int_\Omega \frac{\partial^2 \varphi_{\ell,j}}{\partial t^2} \frac{\partial \varphi_{\ell,j}}{\partial t} + \int_\Omega D_j \left(v^+ + \alpha_\ell \varphi_{\ell,j}\right) D_j \frac{\partial \left(v^+ + \alpha_\ell \varphi_{\ell,j}\right)}{\partial t} = 0$$

i.e.,

$$\frac{d}{dt} \left\{ \frac{1}{2} \alpha_\ell \int_\Omega \left(\frac{\partial \varphi_{\ell,j}}{\partial t} \right)^2 + \frac{1}{2} \int_\Omega \left(D_j \left(v^+ + \alpha_\ell \varphi_{\ell,j}\right)\right)^2 \right\} + \int_\Omega \frac{\partial^2 \varphi_{\ell,j}}{\partial t^2} \frac{\partial v^+}{\partial t} = 0. \qquad (A12)$$

iii) We multiply by β_ℓ equation (A12), sum up the results over $\ell = 1,..L$ and $j = 1,..,J$ and finally add equation (A11), we obtain

$$\frac{dE}{dt} = \int_{\partial\Omega} \left(\frac{\partial g}{\partial t} \right)^2 . \qquad (A13)$$

Integrating once in time, we finally get identity (A9).

Once the energy estimate is obtained, the well-posedness is proved via classical techniques of functional analysis : at first, we rewrite problem (A7) as a variational evolution problem, then we introduce a sequence of standard finite dimensionnal approximation spaces to define approximate problems. These problems reduce to ordinary differential equations for which it is easy to prove existence, uniqueness and regularity of the solutions. Moreover, these approximate solutions satisfy an energy identity analogous to identity A9. If coefficients α_ℓ and β_ℓ are positive, each term occuring in the expression of the energy is bounded. One can then extract converging subsequences of approximate solutions and passing to the limit, one proves the claimed result.

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